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FIRST-YEAR COURSE IN MATHEMATICS

GEOMETRY AND TRIGONOMETRY

BY

K. J. SANJANA, M. A.,

PRINCIPAL AND PROFESSOR OF MATHEMATICS,
SAMALDAS COLLEGE, BHAVNAGAR.

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PREFACE.

This book has been written specially for the use of students reading for the First-Year Course in Arts of the Bombay University, and will be found to cover fully the course in *Geometry and Trigonometry*—given below—laid down in the curriculum.

A summary of the Matriculation course in Geometry is given in Section 1. References to the propositions in this preliminary course are indicated by Arabic numerals, and those to the propositions of the book itself by Roman numerals, placed in square brackets. To render the book self-contained, a few propositions in Geometry outside the University curriculum have been either introduced or repeated from the Matriculation course.

In the latter part of the book a complete account has been given of the "Trigonometry of One Angle." It is hoped that a sufficiently practical treatment of problems has been combined with an accurate presentation of principles, so that the student may have little to unlearn hereafter in studying the higher parts of the subject.

The numerous Exercises have been selected with great care and will, it is believed, illustrate most of the types of questions usually set at examinations. The more important of these have been marked with asterisks; other results of equal importance, particularly in Geometry, will be found in the Miscellaneous Examples, which are intended to be taken up on a second reading. For the purpose of securing increased interest generally and assisting students reading privately, a large number of examples have been solved in the exercises or text. In these solutions stress has been laid on the utility of accurate drawing and graphical work.

The Tables of the Logarithms of Numbers and of the Natural and Logarithmic Functions of Angles have been derived from W. & R. Chambers's *Mathematical Tables* (1885). Some errors of the ordinary Four-figure Tables have been removed, and a Table of Natural Secants has been added.

Corrections for any mistakes left and suggestions for improved treatment will be gratefully received.

K. J. SANJANA

Jan. 1, 1914.

GEOMETRY TREATED IN CONJUNCTION WITH TRIGONOMETRY OF ONE ANGLE.—One Paper.

GEOMETRY.

The questions in Practical Geometry shall be set on the constructions contained in Schedule A, together with easy extensions of them as riders if desired. A candidate should provide himself with a ruler graduated in inches and tenths of an inch and in centimetres and millimetres, a set square, a protractor, compasses and a hard pencil. All figures should be drawn accurately.

The questions on Theoretical Geometry shall consist of theorems contained in Schedule B, together with questions upon these theorems, easy deductions from them and arithmetical illustrations. Any proof of a proposition shall be accepted which forms a part of any systematic treatment of the subject; the order in which the theorems are stated in Schedule B is not imposed as the sequence of the treatment. Proofs which are only applicable to commensurable quantities shall be accepted. The use of intelligible abbreviations is recommended.

SCHEDULE A.

Division of straight lines into parts in any given proportions.

Construction of a triangle or a square equal in area to a given polygon.

Construction of common tangents to two circles.

Simple cases of the construction of circles from sufficient data.

Construction of a fourth proportional to three given straight lines and a mean proportional to two given straight lines.

Construction of a regular pentagon

Description in a given triangle of a triangle similar and similarly placed to another given triangle.

Description of squares in a triangle and in or about a given quadrilateral.

SCHEDULE B.

Proportion: Similar Triangles.

If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally; and the converse. 1

If two triangles are equiangular, their corresponding sides are proportional; and the converse. 3

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

If two triangles have one angle of the one equal to one angle of the other and the sides about another angle of each proportional, the sides 5 opposite the equal angles being homologous, the third angles of the triangles are either equal or supplementary.

The internal bisector of an angle of a triangle divides the opposite 2 side internally in the ratio of the sides containing the angle, and likewise the external bisector externally.

In a right-angled triangle the perpendicular drawn from the right angle to the base will divide the triangle into two parts which are similar to 6 the whole and to each other.

If an angle of a triangle be bisected by a straight line which cuts the opposite side, the sum of the rectangle contained by the two segments of that side and the square on the bisecting line is equal to the rectangle contained by the other two sides of the triangle.

If a perpendicular be drawn from a vertex of a triangle to the opposite side, the rectangle contained by the other sides of the triangle is equal to the rectangle contained by the prependicular and the diameter of the circle described about the triangle.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by the two pairs of opposite sides.

The ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

If two triangles (or parallelograms) have one angle of the one equal to one angle of the other, their areas are proportional to the areas of rectangles contained by the sides about the equal angles.

Concurrency and Collinearity.

If three concurrent straight lines are drawn from the angular points of a triangle to meet the opposite sides, the product of three alternate segments taken in order is equal to the product of the other three segments.

If a transversal is drawn to cut the sides or the sides produced of a triangle, the product of three alternate segments taken in order is equal to the product of the other three segments.

The three medians of a triangle meet in a point, and their common point is a point of trisection of each median.

The three lines drawn through the angular points of a triangle perpendicular to the opposite sides are concurrent.

The three lines which bisect the angles of a triangle are concurrent; and so also are the bisector of one of the interior angles of a triangle and the bisectors of the other two exterior angles.

The three lines drawn through the middle points of the sides of a triangle perpendicular to those sides are concurrent.

In any triangle the three middle points of the sides, the three feet of the perpendiculars drawn from the angular points on the sides, and the three middle points of the lines joining the orthocentre to the angular points all lie on a circle whose diameter is equal to the radius of the circumscribed circle and whose centre is the middle point of the line joining the orthocentre and circumcentre.

If from any point on the circumference of a circle, perpendiculars be drawn to the sides of an inscribed triangle, the three feet of the perpendiculars lie on a straight line.

Harmonic Section.

Division of a given straight line internally and externally so that its segments may be in a given ratio.

The locus of a point whose distances from two fixed points have a constant ratio is a circle.

Centre of Similitude.

If any two unequal similar figures are placed so that their homologous sides are parallel, the lines joining corresponding points in the two figures meet in a point, whose distances from any two corresponding points are in the ratio of any pair of homologous sides.

Every straight line which passes through the extremities of two parallel radii of two fixed circles passes through one or other of two fixed points.

Pole and Polar.

If a straight line be drawn through a given point to cut a given circle, the intersection of the tangents at the two points of section always lies on a fixed straight line.

If one point lie on the polar of another point, the second point lies on the polar of the first point

Radical Axis.

Determination of the locus of points from which tangents drawn to two given circles are equal.

The radical axes of three circles taken in pairs are concurrent.

Construction of the radical axis of two given circles.

Expressions for radii of circumcircle, in-circle and ex-circle of a triangle.

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Trigonometrical functions of positive angles less than two right angles.

The formulæ relating to triangles with radii of circles connected with triangles, formulæ for tan $\frac{A}{2}$, etc.

Four-figure Logarithms.

Solution of triangles with simple two dimensional problems in heights and distances.

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GEOMETRY.

CHAPTER I.

RECAPITULATION. THEOREMS AND PROBLEMS ON RECTILINEAL FIGURES.

§ 1. SUMMARY OF PREVIOUS RESULTS.

A. Theorems.

1. All right angles are equal. (*a*) At a point in a straight line only one perpendicular can be drawn to the line. (*b*) The supplements and complements of equal angles are equal.

2. If a straight line meets another straight line, it makes the adjacent angles equal to two right angles: conversely, if the adjacent angles made by two straight lines at a point of another straight line are together equal to two right angles, the two lines are in one and the same straight line.

3. If a straight line cuts another straight line, the vertically opposite angles are equal: conversely, if the vertically opposite angles made by two straight lines at a point of another straight line are equal, the two lines are in one and the same straight line.

Two triangles are identically equal when—

4. two sides of one are equal respectively to two sides of the other, and the included angles are likewise equal ;

5. three sides of one are equal respectively to three sides of the other ;

6. two angles of one are respectively equal to two angles of the other, and likewise one side of one is equal to the corresponding side of the other.

7. If two triangles have two sides of one equal respectively to two sides of the other, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary ; in the former case the triangles are identically equal. (*a*) If the two given angles are both right or both obtuse, the triangles are identically equal.

8. A triangle which has two equal sides has the opposite angles equal: conversely, a triangle which has two equal angles has the opposite sides equal.

9. If any side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.

10. Any two angles of a triangle are together less than two right angles. (*a*) From a point outside a given straight line only one perpendicular can be drawn to the line.

The converse of (10) cannot be demonstrated without the assumption of some fact about parallel straight lines; it is often assumed as an axiom (*Euclid's*), and stated thus: if two straight lines falling on a third make the two interior angles on the same side less than two right angles, the two lines when produced far enough meet on that side on which are the angles less than two right angles. Or the following axiom (*Playfair's*) may be used instead: through the same point there cannot be drawn two straight lines parallel to a given straight line.

11. If a straight line intersecting two other straight lines makes (i) a pair of alternate angles equal, (ii) a pair of corresponding angles equal, (iii) a pair of interior angles on the same side supplementary, the two given straight lines are parallel.

12. If a straight line intersects two parallel straight lines, it makes (i) the alternate angles equal, (ii) the corresponding interior and exterior angles equal, (iii) the interior angles on the same side supplementary.

13. Straight lines which are parallel to the same straight line are parallel to one another.

14. If a side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.

15. The three angles of a triangle amount to two right angles. (*a*) When one angle of a triangle is right, the other two are complementary. (*b*) When two angles of a triangle are equal to the third, this third angle is right.

16. In a parallelogram, (i) the adjoining angles are supplementary and the opposite angles are equal, (ii) the opposite sides are equal, (iii) each diagonal divides the parallelogram into congruent triangles, and (iv) the diagonals bisect each other.

17. If a quadrilateral has two opposite sides equal and parallel, its other sides are equal and parallel, and its diagonals bisect each other.

18. Two rectangles are congruent when two adjoining sides of one are equal respectively to two adjoining sides of the other. (*a*) Two squares are congruent when a side of one equals a side of the other: conversely, when two straight lines are equal, the squares described on them are equal.

Hence any rectangle is said to be contained by two of its adjoining sides, and any square to be the square of any one of its sides.

19. If two straight lines fall on three or more parallel straight lines and the segments of one made by the parallels are equal, then the segments of the other made by the parallels are also equal. (*a*) The straight line through the middle point of one of the sides of a triangle parallel to a second side bisects the third side: (*b*) conversely, the straight line joining the middle points of two sides of a triangle, is parallel to the third side; and it is also half of the side to which it is parallel.

20. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it; and if two angles are unequal, the greater angle has the greater side opposite to it.

21. The sum of two sides of a triangle is greater than the third side. (*a*) The difference of two sides of a triangle is less than the third side.

22. If two triangles have two sides of one equal respectively to two sides of the other, but have the included angles unequal, then the third side of that which has the greater angle is greater than the third side of the other: conversely, if two sides of one being equal respectively to two sides of the other the third sides are unequal, then the included angle of that triangle which has the greater side is greater than the included angle of the other.

23. Parallelograms on the same base, or on equal bases, and between the same parallels, have equal areas. (*a*) A parallelogram is equal in area to a rectangle whose sides are equal to the base and height of the parallelogram.

24. A triangle is equal in area to half the rectangle whose sides are equal to the base and height of the triangle. (*a*) Triangles on the same base or on equal bases, and between the same parallels, are equivalent: conversely, triangles of equal area on the same side of the same base or of equal bases in a straight line, are between the same parallels.

In the following four results, x, a, b,...are finite straight lines or finite parts of straight lines; and xa, a^2 denote respectively the rectangle contained by x and a, and the square whose side is a.

25. $x(a+b+c+\quad)=xa+xb+xc...$;

26. $(a+b)^2 =a^2+2ab+b^2$;

27. $(a-b)^2 =a^2-2ab+b^2$;

28. $(a+b)(a-b) =a^2-b^2$.

29. In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides about the right angle. (*a*) The square on either side is equal to the difference of the squares on the hypotenuse and the other side.

30. In an obtuse-angled triangle the square on the side opposite to the obtuse angle is greater than the sum of the squares on the sides about the obtuse angle, by twice the rectangle contained by either of these sides and the projection upon it of the other side.

31. In any triangle the square on the side opposite to an acute angle is less than the sum of the squares on the sides about the acute angle, by twice the rectangle contained by either of these sides and the projection upon it of the other side.

32. If the square on one side of a triangle is equal to, greater than, less than the sum of the squares on the other two sides, then the angle opposite to the side is respectively a right, an obtuse, an acute angle.

33. In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects that side.

34. The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two given points.

35. The locus of a point which is equidistant from two fixed intersecting straight lines is the pair of straight lines which bisect the angles between the two given lines.

36. The straight line drawn from the centre of a circle to the middle point of a chord (not passing through the centre) is perpendicular to the chord: conversely, the straight line drawn from the centre perpendicular to a chord bisects the chord.

37. One circle, and only one, can be drawn through any three points not in a straight line. (*a*) Circles which have three points in common coincide altogether. (*b*) If from a point within a circle more than two equal straight lines can be drawn to the circumference, the point is the centre of the circle.

38. In equal circles, arcs subtending equal angles at the centres are equal : conversely, angles at the centres subtended by equal arcs are equal.

39. In equal circles, arcs cut off by equal chords are equal, the minor to the minor and the major to the major: conversely, chords cutting off equal arcs are equal.

40. In equal circles, chords which are equal are equally distant from the centres: conversely, chords which are equally distant from the centres are equal.

41. The tangent at any point of a circle is perpendicular to the radius drawn to the point : conversely, the straight line perpendicular to any radius at its extremity touches the circle at the extremity.

42. Two and only two tangents can be drawn to a given circle from an external point ; these tangents are of equal length, and equally inclined to the straight line joining the external point to the centre.

43. If two circles touch one another, their point of contact is in a straight line with their centres : conversely, if two circles have a common point in a straight line with their centres, they touch one another at the point.

44. Any angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

45. Angles in the same segment of a circle are equal: conversely, if the straight line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

46. The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is acute, and that in a segment less than a semicircle is obtuse. (*a*) The locus of a point at which a given finite straight line subtends a right angle is the circle described on the given line as diameter.

47. The opposite angles of any quadrilateral inscribed in a circle are supplementary: conversely, if two opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

48. Each of the angles between a tangent to a circle and a chord drawn from its point of contact, is equal to the angle in the alternate segment of the circle: conversely, if a straight line through an extremity of a chord of a circle make an angle with it equal to that in the alternate segment of the circle, the straight line touches the circle.

B. Problems.

49. To bisect (i) a given angle, (ii) a given finite straight line.

50. To draw a perpendicular to a given straight line from a given point on the line or outside the line.

51. To make at a given point of a given straight line an angle equal to a given angle.

52. To draw through a given point a straight line parallel to a given straight line.

53. To construct a triangle, given (i) three sides, (ii) two sides and the included angle, (iii) two angles and the adjoining side or the side opposite to one of them, and (iv) two sides and the angle opposite to one of them.

54. To divide a given straight line into a given number of equal parts.

55. To construct a square upon a given finite straight line.

56. To find the centre of a given circle or of a given arc of a circle.

57. To bisect a given arc of a circle.

58. To draw a tangent to a circle from a given point upon or outside the circumference.

59. To describe a square in or about a given circle.

60. To describe a circle in or about a given square.

61. To describe in or about a given circle a triangle equiangular to a given triangle.

62. To describe a circle in or about a given triangle.

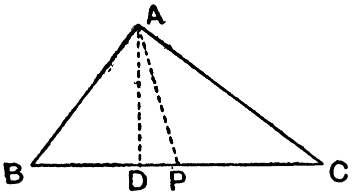
63. To describe on a given straight line a segment of a circle containing a given angle.

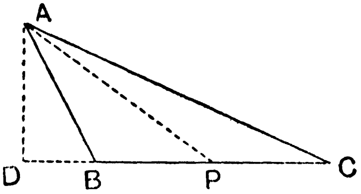
§ **2.** The following theorems and the problems of § 3 are supplemental to those given above. They will often be required in the proof of subsequent propositions and in the construction of equivalent rectilineal figures.

Def. The straight line joining the middle point of a side of a triangle to the opposite vertex is called the *median* to that side from the vertex.

Proposition I. Theorem.

In any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the third side and the projection upon this side of the median drawn to it.





Given that AP is a median and DP its projection on the side BC of a △ ABC : to prove that $AC^2 - AB^2 = 2\ BC.DP$.

Draw AD perp. to BC ; and let ∠ APB be acute, so that ∠ APC is obtuse.

Then $AB^2 = BP^2 + AP^2 - 2BP.DP$, [31]

and $AC^2 = PC^2 + AP^2 + 2PC.DP$. [30]

Hence, ∵ $BP^2 = PC^2$, we have by subtraction

$$AC^2 - AB^2 = 2DP\ (PC + BP) = 2BC.DP.$$

If ∠ APB is obtuse, we shall have $AB^2 - AC^2 = 2BC.PD$.

By adding up the values of AB^2 and AC^2, we get $AB^2 + AC^2 = 2\ (BP^2 + AP^2)$. See [33].

Exercises.

1. Draw a triangle ABC in which BC = 7, CA = 6, AB = 5 ; bisect AB in R and draw CF perpendicular to it. Measure the length FR ; also find it by calculation.

2. The sides of a triangle are 3·6, 3, 2·5 cm : find the median to the largest side, and verify the result by calculation.

*3. The difference of the squares of the distances of a moving point from two fixed points is invariable : prove that the point moves on a straight line.

4. The sum of the squares of the distances of a moving point from two fixed points is invariable : prove that the point moves on a circle, and find the centre and radius of the circle.

5. Given the lengths of the medians of a triangle; obtain formulæ for calculating the sides.

6. The sides of a triangle are 2, 2·4, 3·5 in.: find the segments of the mean side made by the perpendicular from the opposite angle. Also, solve the problem when the length of the third side is 3·1 in.

---

*7. In any triangle ABC, $BC=a$, $CA=b$, $AB=c$: prove that the square of the perpendicular from A on BC is

$$\frac{(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)}{4a^2}.$$

Solution. Taking the first figure of the proposition,

$$AC^2-AB^2=2BC.DP\,;\quad \therefore\quad DP=\frac{b^2-c^2}{2a}:$$

$$\text{and } PC=\frac{a}{2}=\frac{a^2}{2a}\,;\quad \therefore\quad CD=PC+DP=\frac{a^2+b^2-c^2}{2a}.$$

$$\begin{aligned}
\text{Hence } AD^2&=AC^2-CD^2 && [29]\\
&=(AC-CD)\,(AC+CD) && [28]\\
&=\left(b-\frac{a^2+b^2-c^2}{2a}\right)\left(b+\frac{a^2+b^2-c^2}{2a}\right)\\
&=\frac{c^2-a^2-b^2+2ab}{2a}\cdot\frac{a^2+b^2+2ab-c^2}{2a}\\
&=\frac{c^2-(a-b)^2}{2a}\cdot\frac{(a+b)^2-c^2}{2a} && [27 \text{ and } 26]\\
&=\frac{(a+b+c)\,(a+b-c)(c+a-b)(c-a+b)}{4a^2}. && [28]
\end{aligned}$$

*8. Shew that the area of a triangle in terms of the lengths of its sides a, b, c, is $\frac{1}{4}\sqrt{\{(a+b+c)\,(a+b-c)\,(c+a-b)(b+c-a)\}}$.

9. The sides of a triangle are (i) 12, 15, 9, (ii) 3, 7, 8: find its area by help of the preceding exercise. Verify the results by geometrical construction.

10. If in a quadrilateral the sums of the squares on opposite sides are equal, then the diagonals are at right angles.

11. If a quadrilateral ABCD is such that $AB^2+BC^2=CD^2+DA^2$, then the projections of its opposite sides on BD are equal.

*12. AP, BQ, CR are the medians of a triangle ABC: prove that $3\,(BC^2+CA^2+AB^2)=4\,(AP^2+BQ^2+CR^2)$.

13. Any point O is taken in the plane of a rectangle ABCD: prove that $OA^2+OC^2=OB^2+OD^2$. Conversely, if ABCD is a parallelogram and $OA^2+OC^2=OB^2+OD^2$, then the parallelogram is a rectangle.

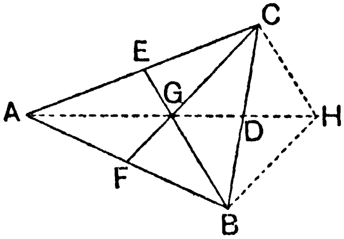
*14. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals. Conversely, if a quadrilateral ABCD is such that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2$, then the quadrilateral is a parallelogram.

15. The side AB of an equilateral triangle ABC is produced to D so that $AD=3\,AB$: prove that $CD^2=7AB^2$.

16. A, B are two given points outside a given straight line CD: find a point P in CD such that the difference of the squares on PA, PB may be equal to twice the square on AB.

Proposition II. Theorem.

The three medians of a triangle meet all in one point, which is a point of trisection of each of them.



Let BE, CF be the medians through the vertices B, C of a △ ABC, meeting in G; join AG, and produce it to meet BC in D: to prove that AD is the median through A, and that G trisects each median.

Produce AG to H, make GH = AG, and join BH, CH.

∵ AF = FB (*Hyp.*), AG = GH (*Cons.*),

∴ $FG = \frac{1}{2}$ BH, and FGC and BH are parl., [19, *b.*]

so also $EG = \frac{1}{2}$ CH, and EGB and CH are parl.

Hence BGCH is a parlgm., and the diagonals BC, GH bisect one another. [16, iv.]

∴ BD = DC, and AGD is the median through A.

∴ the three medians are concurrent.

Again ∵ $FG = \frac{1}{2}$ BH and BH = GC, [16, ii.]

∴ $FG = \frac{1}{2}$ GC $= \frac{1}{3}$ FC: so also $EG = \frac{1}{3}$ EB.

And $DG = \frac{1}{2}$ GH $= \frac{1}{2}$GA, ∴ $DG = \frac{1}{3}$ DA;

∴ G is a point of trisection of each median.

Def. When three or more straight lines pass through a point, they are said to be *concurrent*.

Def. When three or more points lie on a straight line, they are said to be *collinear*.

Def. The point of concurrence of the medians of a triangle is called the *centroid* of the triangle.

Exercises.

17. If DF, DE, EF be joined in the figure of the Prop., then G is the centroid of the triangle DEF.

18. ABC is a triangle and G is the centroid: prove that

(i) $AB^2+AC^2=BG^2+CG^2+4AG^2$,

(ii) $AG^2+BG^2+CG^2=\frac{1}{3}(BC^2+CA^2+AB^2)$.

---

19. Construct a triangle, given the lengths of (i) two sides and one median, (ii) two medians and one side.

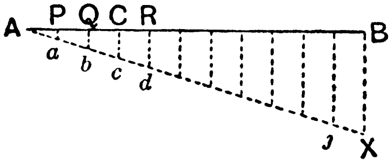
*20. Given the lengths of the three medians, shew how to construct the triangle.

21. A triangle is formed with the medians AD, BE, CF of a given triangle ABC for its sides: prove that each median of this triangle is $\frac{3}{4}$ of the corresponding side of the triangle ABC.

§ 3. PROBLEMS AND RECTILINEAL CONSTRUCTIONS.

Proposition III. Problem.

To divide a given finite straight line into two parts in any given ratio.



Let AB be the given straight line: it is required to divide it into two parts in a given ratio, say that of the numbers 3 and 8.

Draw the straight line AX making any angle with AB. Divide AX into 11 (= 3 + 8) equal parts at the points *a*, *b*, *c*, ... *j* [54]; join BX, and draw *c*C parl. to BX to meet AB: then AB shall be divided at C as required.

Draw *a*P, *b*Q, *d*R, ... parl. to BX.

Then AB is divided into 11 equal parts at P, Q, C, R, . . [19] of which AC contains 3 and CB contains 8.

∴ AB is divided at C in the ratio of 3 to 8.

Similarly, if the ratio is $a : b$, we divide AX into $a+b$ equal parts, join the last point of division to B, and draw through the point where a parts from A are terminated a line parallel to the joining line. If the given ratio is between numbers both of which are not integral, it should be first reduced to the ratio of two whole numbers by arithmetical methods.

Def. The *perimeter* of any rectilineal figure is the sum of all its bounding sides.

Exercises.

22. Divide a given straight line of length 1·7″ into two parts in the ratio of 2 to 7, and measure the length of each part.

23. Divide a straight line 3·7 cm. long into two parts of which one shall be (i) one-eighth of the whole, (ii) one-eighth of the other part.

24. In a given straight line AB find a point C such that AC may have to AB the ratio of m to $m+n$.

25. Prove the following construction for trisecting a given line AB. — Through B draw any straight line XBY, make BY = XB, join XA; bisect XA in D, and join DY to cut AB in C: if AC is bisected in E, E and C are the points of trisection.

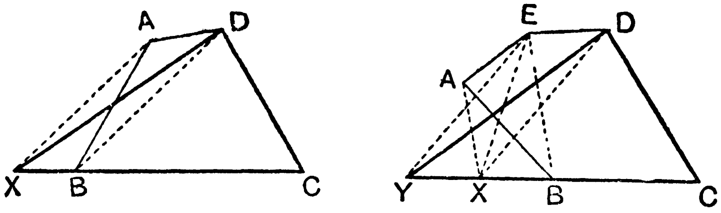
*26. Shew how to divide a given straight line into three or more parts whose ratios are known.

27. Draw any triangle the lengths of whose sides shall be as 3, 4, 6.

28. The perimeter of a triangle is 57 mm.; the second side is half as much again as the first, and the third is half as much again as the second. Construct the triangle, and verify the lengths of the sides by calculation.

Proposition IV. Problem.

To construct a triangle equal in area to a given rectilineal figure.



First let the given figure be the quadrilateral ABCD: it is required to construct a triangle of equal area.

Join DB; draw AX parl. to DB to meet CB produced in X, and join DX. Then DXC shall be the required triangle.

For $\triangle$ XBD = $\triangle$ ABD; [24, *a*]
add the $\triangle$ BCD to each:

∴ $\triangle$ XCD = quad. ABCD.

Next let the figure be the pentagon ABCDE.

Join EB; draw AX parl. to EB to meet CB produced in X, and join EX.

Then $\triangle$ XBE = $\triangle$ ABE;
add the quad. BCDE to each:

∴ quad. XCDE = pentagon ABCDE.

Now as in the first case the quad. XCDE can be reduced to the equivalent $\triangle$ YCD, which therefore is equal in area to the given pentagon.

The process may be similarly applied to rectilineal figures having a larger number of sides.

Rectilineal figures of equal area are often called *equivalent.* To find the area of any polygon we should first reduce it to an equivalent triangle: as this triangle has half the area of the rectangle having the same base and height [24], the area of the polygon is obtained by finding that of the rectangle by the usual arithmetical rule.

Def. A rectilineal figure which has all its angles equal to each other and all its sides equal to each other is said to be a *regular figure.*

Exercises.

29. Construct a quadrilateral having three sides 18, 32, 21 mm. in succession, the angles between the side of 32 mm. and the adjacent sides being 60^0 each. Reduce the figure to an equivalent triangle.

30. Construct a parallelogram with one angle of 45^0 and equal in area to the quadrilateral in the preceding exercise.

31. Construct any rectangle equal in area to a pentagon ABCDE in which

$$AB=7,\ BC=5,\ CD=6,\ DE=4{\cdot}5,\ EA=6;$$
$$\angle ABC=110^0,\ \angle BCD=118^0.$$

*32. Construct a regular hexagon of side 1·5 cm., reduce it to a triangle and thence find its area. Verify the result by obtaining an exact formula for the area of the hexagon.

---

33. A pentagon is formed by a square of side ·8″ surmounted by an equilateral triangle of the same side: reduce the figure to an equivalent triangle and thence find its area. Also obtain an exact expression for the area and verify the measurement.

34. By means of the protractor construct a regular octagon of side 1 cm : reduce it to an equivalent triangle.

35. Construct a triangle equal in area to the sum of two triangles with sides 2·4, 2, 1·8, and 2·2, 1·6, 1·4, and thence find the sum of their areas.

36. Through a given point in one side of a given triangle draw a straight line which shall cut off $\frac{1}{2}$ or $\frac{1}{8}$ of the area of the triangle.

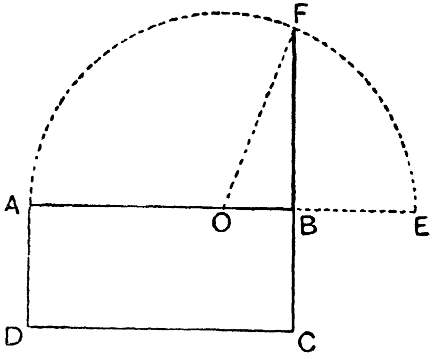
*37. Bisect the area of a quadrilateral by a straight line drawn through one of its angular points.

38. Construct a triangle equivalent to a given triangle ABC, having one angle equal to the $\angle BAC$ and one side equal to a given straight line.

39. Draw a quadrilateral ABCD in which $AB=4$, $BC=8$, $CD=8$, $DA=6{\cdot}5$, $\angle ABC=98^0$, and divide the figure into four equal parts by straight lines drawn from D.

Proposition V. Problem.

To construct a square equal in area to a given rectangle.



Let ABCD be the given rectangle: it is required to construct an equivalent square.

Produce AB to E, making $BE = BC$; bisect AE in O, and on AE as diameter describe the semicircle AFE. Produce CB to meet the semicircle in F: then BF is a side of the required square.

Join OF. $\because BF^2 = OF^2 - OB^2$ [29]

$$= OA^2 - OB^2$$

$$= (OA + OB)(OA - OB) \quad [28]$$

$$= (OA + OB)(OE - OB)$$

$$= AB.BE = AB.BC;$$

$\therefore$ the sq. on BF is equal in area to the rect. ABCD.

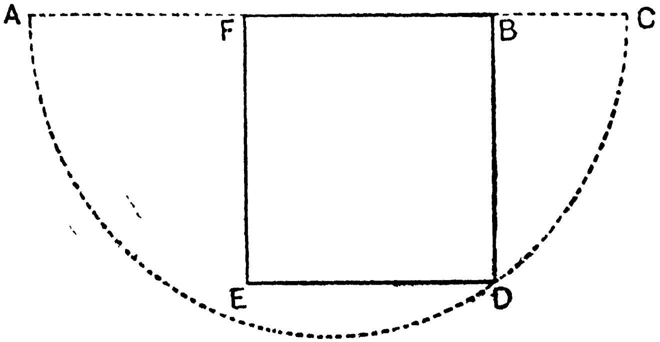
Cor. The square on the perpendicular drawn from any point on the circumference of a circle to any diameter of the circle equals the rectangle contained by the segments of the diameter.

The area of a square equal to a given rectangle will not in general be represented by a square number; hence the side of such a square will often be an incommensurable *quadratic surd*, as explained in Algebra. The process given above may therefore be taken as a geometrical method for constructing such surds. Thus, to find $\sqrt{(30)}$, we may draw a rectangle of sides 6 and 5, or 10 and 3, and then find the side of the equivalent square; the measure of this side will give $\sqrt{(30)}$ approximately. We can also obtain such surds by repeated application of Pythagoras' theorem. Thus, taking the same example—as $30 = 25 + 4 + 1$, $\sqrt{(29)}$ will be the hypotenuse of a right-angled triangle whose sides are 5 and 2, and $\sqrt{(30)}$, of another whose sides are $\sqrt{(29)}$ and 1; hence $\sqrt{(30)}$ may be constructed and measured.

Exercises.

40. Construct a rectangle of sides 12 and 3·5, and draw the equivalent square : measure its side and verify by calculation.

Solution. Scale, unit = ·2 in. Take AB = 2·4 in., BC (in the same line) = ·7 in. Describe the semicircle on diameter AC, and let BD perp. to AC meet it in D. Then BD is the side of the required square, BDEF.



On measurement, we find BD = 1·3 in. = 6·5 units. The arithmetical value of the side is $\sqrt{(12 \times 3{\cdot}5)} = \sqrt{(42)} = 6{\cdot}48$ nearly.

41. Draw any rectangle of area 40·8 sq. cm. and find the side of the equivalent square : verify by calculation.

42. Find by geometrical construction a side of the square equivalent to an equilateral triangle of side 1·2 in.

*43. Describe a rectangle equivalent to a given square, and such that the sum of two adjacent sides is equal to a given straight line. When is the construction not possible?

*44. Describe a rectangle equivalent to a given square, and such that the difference of two sides is equal to a given straight line.

45. Solve the following equations geometrically, giving reasons for the constructions adopted :—

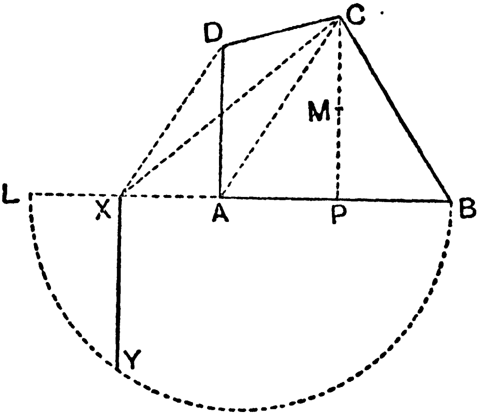
(i) $x^2 - 12x + 35 = 0$, (ii) $x^2 - 12x - 28 = 0$, (iii) $x^2 - 5x + 3 = 0$.

46. Construct the surds $\sqrt{(19)}$ and $\sqrt{(35)}$ by means of Prop. V, and $\sqrt{(13)}$ and $\sqrt{(54)}$ by means of Pythagoras' Theorem.

*47. Prove that the perimeter of a square is less than the perimeter of any rectangle of equal area.

Proposition VI. Problem.

To construct a square equivalent to any given rectilineal figure.



Let ABCD be the given rectilineal figure: it is required to construct an equivalent square.

Construct the Δ XBC of area equal to that of ABCD (IV); draw CP perp. to AB and bisect it at M. Produce BX to L, making XL = MP; on BL as diameter describe a semicircle, and draw XY perp. to BL to meet the ⊙ce: then XY is the side of the sq. required.

For $XY^2 = XB.XL$ [V, *Cor.*]

$= \frac{1}{2}XB.CP = \Delta XBC$ [24]

= quad. ABCD.

The method is similar for figures of more than four sides.

Exercises.

48. Find the side of the square equivalent to a quadrilateral ABCD in which AB = 11, BC = 9, CD = 6, DA = 8 and BD = 10. If the measures are in tenths of an inch, find the area in square inches.

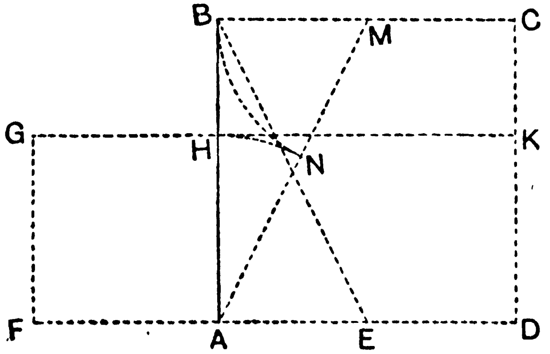
49. On a base 15 mm. long describe a regular pentagon by using the protractor: obtain and measure the side of the equivalent square.

50. A regular hexagon is 1″ in side; find its area by reducing it to a square, and verify by obtaining a formula for the area of the hexagon.

51. Draw the quadrilateral ABCD from the following data:— AB = 25 mm., BC = 18 mm., ∠ ABC = 70°, ∠ BCD = 140°, ∠ DAB = 65°. Construct accurately a square of equal area.

Proposition VII. Problem.

To divide a given straight line into two parts such that the rectangle contained by the whole line and one part is equal to the square on the other part.



Let AB be the given straight line: it is required to divide it at H so that the rect. AB.HB is equal to the sq. on AH.

On AB describe the sq. ABCD; bisect AD in E, join EB; with E as centre and EB as radius draw a circle cutting DA produced in F. With A as centre and radius AF draw a circle cutting AB in H: then H is the required point of section.

Through F and H draw the lines FG and GHK respectively parl. to AB and AD: then AFGH is a sq.

Now rect. DF.FG = rect. DF.AF

$$= (EF + EA)(EF - EA)$$

$$= EF^2 - EA^2 \qquad [28]$$

$$= EB^2 - EA^2$$

$$= AB^2.$$

∴ fig. DABC = fig. DFGK; taking away the fig. DH, we have fig. KHBC = fig. AFGH, that is, rect. BC.HB = sq. AFGH.

∴ rect. AB.HB = AH^2.

Def. When a straight line is so divided that the rectangle contained by it and one part equals the square on the other part, it is said to be divided in *medial section* (or in *extreme and mean ratio*).

It is seen that AH = AF = EF − AE = EB − AE : hence the following construction may be given instead of that in the proposition. At B in AB erect the perpendicular BC ; take BM = $\frac{1}{2}$ AB ; join MA, and cut off MN = MB. Aso cut off AH = AN : then H is the required point of section. For, AH = AM − MN = EB − EA, as the Δs AMB, AEB are congruent.

If the length of AB is denoted by $2a$, that of MB is a, so that AM $=\sqrt{5}a$, and $\therefore$ AH $=\sqrt{5}a-a=(\sqrt{5}-1)a$. It follows that BH $=2a-(\sqrt{5}-1)a=(3-\sqrt{5})a$; and it can be readily proved that AB.HB $=2(3-\sqrt{5})a^2=(6-2\sqrt{5})\,a^2=(\sqrt{5}-1)^2a^2=$ AH^2.

Again, denoting AH by x, we get from AB.HB = AH^2 the equation $2a\,(2a-x)=x^2$ or $x^2+2ax-4a^2=0$. It is found that the positive root of this quadratic equation is $(\sqrt{5}-1)a$. We might therefore solve the problem by first solving this equation and then constructing for the length $(\sqrt{5}-1)a$. The negative root is $-(\sqrt{5}+1)a$.

The numerical value of $\sqrt{5}-1$ is very nearly 1·236 ; thus AH is nearly ·62 of AB and HB is ·38 of AB.

Exercises.

52. Adopting the construction given in the Note, prove that AH^2 = AB.HB from the triangle ABM without completing the squares.

53. Divide a straight line of length 28 mm. in medial section: measure the segments and verify the measurement by the solution of a quadratic equation.

54. Prove that a straight line divided medially is also divided in such a manner that (i) the difference of the squares of the segments = the rectangle contained by them, (ii) the rectangle contained by the line and the larger segment = the rectangle contained by the smaller segment and the sum of the line and the larger segment.

*55. A straight line AB is divided in medial section at X : (i) if from AX the greater segment a part AY = XB is cut off, then AX is divided at Y in medial section ; (ii) if BA is produced to C so that AC = AB, CX is divided at A in medial section.

56. With the notation of the preceding exercise prove that AB^2 − XY^2 = 4AB.XY.

57. With the figure of the Proposition prove that AB^2 + BH^2 = 3AH^2. Verify this result by employing the surd values of the segments.

58. In the same figure, if AB is produced to K so that BK = HB, prove that AK^2 = 5 AH^2.

Miscellaneous Examples I.

1. Construct straight lines representing $\sqrt{6}$ and $\sqrt{7}$ cm. and measure them in mm.

2. Draw a square of side 2·5 cm., and construct another with area $\frac{1}{5}$ of that of the given square. Measure the side and check the result by calculation.

3. A square of side ·6 in. is given: draw another whose area is $\frac{3}{8}$ or $\frac{8}{3}$ of that of the given square.

4. Construct a triangle equivalent to a given quadrilateral and having a base equal to a given straight line.

5. A farmer wishes to fence off a rectangular field two acres in area : find the least length of the fencing required.

6. Construct the quadrilateral ABCD from the following data :—$AB = 20$, $BC = 24$, $CD = 17$, $\angle ABC = 100^\circ$, $\angle BCD = 60^\circ$. Find the equivalent triangle with vertex C and base lying along AB ; and shew how to cut off the third part of the quadrilateral by a line drawn through C.

7. Construct a rectangle equal in area to a given square, and having one of its sides equal to twice or thrice the other side.

*8. The sum of the squares on the sides of any quadrilateral is equal to the sum of the squares on the diagonals together with four times the square on the line joining the midpoints of the diagonals.

*9. In the base BC of a triangle ABC a point P is taken such that BP is $\frac{1}{n}$ of BC : prove that $(n-1)\,AB^2 + AC^2 = \frac{n-1}{n}BC^2 + nAP^2$.

Solution. In the fig. of Prop. I. let $BP = \frac{1}{n}\,BC$.

As proved there, $AB^2 = BP^2 + AP^2 - 2BP.DP$,
and $AC^2 = PC^2 + AP^2 + 2PC.DP$.

Multiply the first by $n-1$ and add up ; we get

$(n-1)\,AB^2 + AC^2 = (n-1)\,BP^2 + PC^2 + nAP^2 + 2PD\{PC - (n-1)BP\}$.

But $BP = \frac{1}{n}BC$, $\therefore$ $CP = \frac{n-1}{n}BC = (n-1)BP$; hence by reduction,

$$(n-1)\,AB^2 + AC^2 = \left(\frac{n-1}{n^2} + \frac{\overline{n-1}^{\,2}}{n^2}\right)BC^2 + nAP^2$$

$$= \frac{n-1}{n}BC^2 + nAP^2.$$

*10. If G is the centroid of a triangle ABC and P any point in the plane of the triangle, prove that

$$AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2.$$

11. The sum of the squares of the distances of a point from the vertices of a given triangle is least when the point is at the

12. The diagonals of a quadrilateral ABCD meet in E, and F is the midpoint of the straight line joining their middle points : prove that $AE^2+BE^2+CE^2+DE^2=AF^2+BF^2+CF^2+DF^2+4EF^2$.

13. AD, BE, CF are the perpendiculars to the sides BC, CA, AB of a triangle ABC, and K, L, M are respectively their middle points : prove that of the three rectangles BC.DK, CA.EL, AB.FM, one is equal to the sum of the other two.

*14. A point P within a triangle ABC moves so that the triangles PAB, PAC are equivalent : prove that the locus of P is a straight line. Hence shew how to find a point Q such that the areas QAB, QBC, QCA are all equal.

15. Prove the following construction for dividing a given straight line AB in medial section.—Produce AB to ′, making $BA'=AB$; bisect BA' in M; draw BN at right angles to AA' and equal to AB; join MN, and from MA cut off $MC=MN$: then AB is divided at C so that $AB\ AC=BC^2$.

*16. Given a straight line AB, find a point H in BA produced such that $AB\ HB=AH^2$.

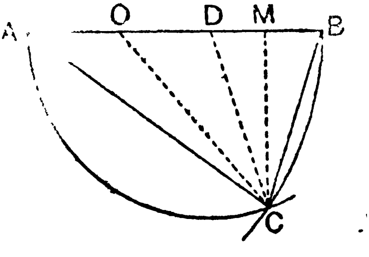
17. Produce a given straight line so that the sum of the squares on the given line and the part produced may be double of the rectangle contained by the whole line produced and the part produced.

18. By obtaining quadratic equations and constructing for their surd roots, solve the following problems geometrically. —Divide a given straight line AB into two parts at H so that the rectangle $AB.HB=\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$ of AH^2.

19. Solve the following equations graphically :—

$$x^2-5x+4=0,\ x^2+5x-4=0.$$

*20. A given straight line AB is divided at D so that $AB.DB=AD^2$; with centres A and D and radii AB and DA respectively circles are drawn meeting in C : prove that in the triangle ABC $\angle B = \angle C = 2\angle A$.



Solution. Bisect AD in O, and draw CM perp to AB. Then $AC^2-CD^2=2\ AD\ OM$ [1].

Now $AC=AB$ and $CD=AD$; $\therefore$ $AC^2-CD^2=AB^2-AD^2=AB^2-AB\ DB$ (*Hyp.*) $=AB\ AD$. Hence $2AD\ OM$ and $AB\ AD$ are equal, $\therefore$ $AB=2OM=2OD+2DM=AD+2\ DM$;

$\therefore$ $DB=2DM$, and DB is bisected at M.

$\therefore$ the Δs CDM, CBM are congruent [4]; $\therefore$ $\angle CBD=\angle CDB$.
But as $DC=DA$, $\angle CDB=2\angle CAD$. [8 and 14]

$\therefore$ $\angle CBD$, as also $\angle ACB$ equal to it, is double of $\angle CAB$.

CHAPTER II.

RECTANGLES OF SEGMENTS OF CHORDS OF A CIRCLE. RADICAL AXIS.

§ 4. ALGEBRAICAL SIGNS OF LINEAR SEGMENTS.

The student is already acquainted with the application of the signs + and − to the lengths of finite straight lines. He knows, for instance, how in graphical representation, the abscissa of a point is reckoned positive or negative according as the point lies in one or the other of two directions from the origin along the axis.

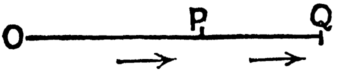


Fig. 1.

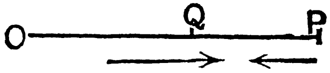
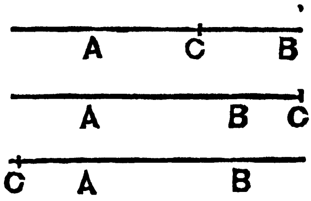


Fig. 2.

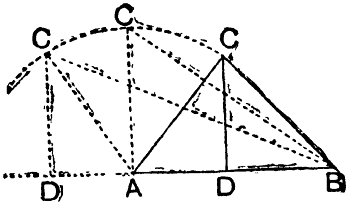
If O is the origin, or the point from which distances are measured, in a line OPQ, we can reach Q from O by making two steps, one from O to P, the other from P to Q. In Fig. 1, where these are in the same direction, we have *step* OQ = *step* OP + *step* PQ, and as OQ is greater than OP we conclude that OP and PQ have the same sign. But in Fig. 2, although we have algebraically *step* OQ = *step* OP + *step* PQ, yet as OQ is less than OP we have to take OP and PQ to be of opposite signs, and only then will the final arithmetical value of OQ be obtained. It is immaterial which of the two directions of motion in the line (indicated by arrow-heads) is taken as positive; but once we have decided on this, the other direction will have to be taken as negative. If the second step is from P back to O, the final distance of Q from O is zero, so that $OP + PO = 0$ or $PO = -OP$. It should be remembered that in such equations a line is to be measured in the direction which is given by the sequence of the letters denoting the line.

Def. When any point C is taken in the indefinite straight line joining two points A and B, the distances AC, CB are called the *segments* or *parts* of the line AB made by C.



It will be seen that in all cases AB is the *algebraic sum* of the two *parts* AC, CB Thus in the adjoined second case CB is negative, in the third case AC is negative. When the origin changes the parts will however change, not only in magnitude but also generally in sign.

Whenever a geometrical figure admits of continuous variation, it will be found that the lengths of some of the distances alter, and in passing from positive to negative values they go through the value zero.

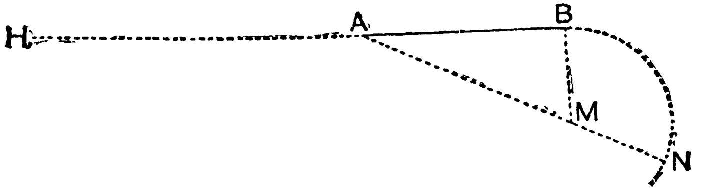


Illustrations. 1. A triangle ABC is taken with an acute angle at A, and the angle is continuously increased so as to become first a right angle and then obtuse. The projection, AD, of AC on AB is found to be positive at first, then to vanish when AC is at right angles to AB (so that A and D coincide), and finally to become negative when the ∠ A increases to an obtuse angle. The following three theorems correspond to these cases, and are the expressions of one and the same algebraical fact :—

AD *pos.*, $BC^2 = AB^2 + AC^2 - 2AB.AD$; AD *zero*, $BC^2 = AB^2 + AC^2$; AD *neg.*, $BC^2 = AB^2 + AC^2 + 2AB.DA$.

2. To divide a given straight line internally and externally into parts in the ratio of 3 to 2. In the second case the smaller part is negative, as $3 - 2 = 1$.

3. To divide a given straight line AB externally at H so that rect. $AB.HB = AH^2$.



At B erect BM perp. to AB and $= \frac{1}{2}$ AB; join MA, and cut off MN = MB from AM produced ; take AH = AN in BA produced.

If $AB = 2a$, $BM = a$ and $AM = a\surd 5$; $\therefore$ $AH = AN = a(\surd 5 + 1)$.

Thus $BH = a(\surd 5 + 3)$; and $AB.HB = 2a^2(3 + \surd 5) = a^2(6 + 2\surd 5)$ $= a^2(\surd 5 + 1)^2 = AH^2$.

In this case AB is said to be *externally divided in medial section* ; and the process above gives the numerical value of the negative root of the quadratic equation obtained in the *Note on Prop.* VII.

4. If A, B, C, P are any points in a straight line, prove that $BC.PA + CA.PB + AB.PC = 0$, and state the arithmetical equation when the points are on the line in the order B A C P

§ 5. RECTANGLE OF SEGMENTS OF A CHORD OF A CIRCLE.

Proposition VIII. Theorem.

If a chord of a given circle passes through a fixed point, the rectangle contained by its segments is constant for all positions of the chord.

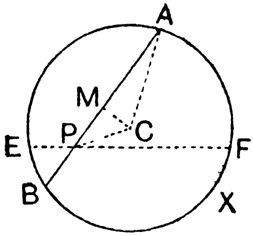


Fig. 1.

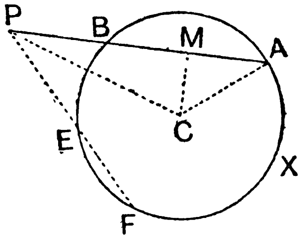


Fig. 2.

In a ○ ABX (centre C) let any chord AB be drawn through a fixed pt. P : to prove that the rect. contained by its segments PA, PB, is constant whatever be the position of the chord.

Bisect AB in M, join CM; then CM is perp. to AB. [36]

Join CP, CA.

Now in Fig. 1, rect.
$$\begin{aligned} PA.PB &= (MA + PM)(MB - MP) \\ &= (MA + PM)(MA - MP) \\ &= MA^2 - MP^2 && [28] \\ &= MA^2 + MC^2 - (MP^2 + MC^2) \\ &= CA^2 - CP^2; && [29] \end{aligned}$$

and in Fig. 2, rect.
$$\begin{aligned} PA.PB &= (PM + MA)(MP - MB) \\ &= (PM + MA)(PM - MA) \\ &= MP^2 - MA^2 && [28] \\ &= MP^2 + MC^2 - (MA^2 + MC^2) \\ &= CP^2 - CA^2. && [29] \end{aligned}$$

But CA and CP are fixed, being respectively the radius of the given ⊙ and the distance of the fixed pt. from its centre.

∴ rect. PA.PB is constant in all cases.

Cor. 1. If two chords of a circle intersect in a point, the rectangle of the segments, made by the point, of one chord equals the rectangle of the segments of the other: for if EF is another chord through P,

$$\text{rect. } PE.PF = \text{difference of } CE^2 \text{ and } CP^2$$
$$= \text{difference of } CA^2 \text{ and } CP^2.$$

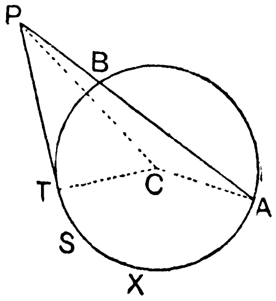
Cor. 2. Conversely, if two finite straight lines intersect in a point so that the rectangle of the segments of one line made by the point equals the rectangle of the segments of the other, their four extremities are concyclic.

Cor. 3. If $r =$ the given radius and $d =$ the distance of the fixed point from the centre, the constant numerical value of the rectangle is $r^2 - d^2$ or $d^2 - r^2$ according as the point is inside or outside the circle.

If we measure distances along AB from the fixed point P, the algebraical value of the rectangle is $d^2 - r^2$ in both cases, as PA.PB is negative in Fig. 1 and positive in Fig. 2. This value is often called *the power of the point* P *with regard to the circle* ABX.

Proposition IX. Theorem.

If from an external point a secant and a tangent are drawn to a circle, the square on the tangent equals the rectangle contained by the whole secant and its part outside the circle: conversely, if the square on a line drawn to a circle from an external point equals the rectangle contained by any whole secant from the point and its part outside the circle, the line drawn to the circle is a tangent.



From a point P outside a ○ ABX let a secant PBA and a tangent PT be drawn to the ○ to prove that rect. $PA.PB = PT^2$.

Join the centre C of the ○ to A, P, T.

∵ $\angle CTP =$ rt. angle [41], ∴ $CP^2 = CT^2 + PT^2$.

∴ $PT^2 = CP^2 - CT^2 = CP^2 - CA^2$

= rect. PA.PB [VIII, *Cor* 3]

Conversely, from P let a secant PAB and a line PT be drawn to the ○, and let $PT^2 =$ rect. PA.PB: to prove that PT touches the ○

If PT does not touch the ○ it will meet it in a point S distinct from T.

∵ PBA, PTS are two secants drawn through P,

∴ rect. PT.PS = rect. PA.PB. [VIII, *Cor.* 1]

$= PT^2$, [*Hyp.*]

∴ PS = PT, which is impossible unless S coincides with T.

∴ PT does not meet the ○ in a point other than T, that is, PT touches the ⊙ at T.

The result can also be obtained by turning the line PAB about P until the points of intersection coincide; when A and B are both at T, PAB becomes the tangent PT and rect. PA.PB becomes PT^2. When P is within the circle, PA and PB are of unlike sign, the rectangle is negative and no single line like PT can exist: in this case, however, when AB becomes perpendicular to CP, AB is bisected at P, and the constant value of the rectangle equals the square of half the chord AB.

Def. Two circles are said to *intersect orthogonally* or *at right angles* when their tangents at a point of section are at right angles. In this case each tangent passes through the centre of the other circle, and the square of the distance between the centres equals the sum of the squares of the radii.

Exercises.

*59. P is any point on the circumference of a circle of diameter AB, and PM is the perpendicular on AB: prove that $PM^2 = AM \cdot MB$, and $PA^2 = AM.AB$.

*60. In a right-angled triangle the perpendicular is drawn from the right angle to the hypotenuse: prove that the square on either side of the triangle is equal to the rectangle contained by the hypotenuse and its segment adjacent to the side.

61. Divide a given straight line internally and externally into two parts whose rectangle may be equal to a given square.

62. Produce a given straight line so that the rectangle contained by the given line and the produced part may be equal to the area of a given polygon.

*63. XY is a fixed straight line perpendicular to the diameter AB of a given circle; a straight line through A meets the circle in P and XY in Q. Prove that rect. AP.AQ is constant.

*64. From a fixed point A a variable line AP is drawn to any point P of a given straight line XY; a point Q is taken on AP so that the rect. AP.AQ is constant. Find the locus of Q.

65. Two chords of a circle intersect at right angles within the circle; the two segments of one chord are 3 and 7 cm., and one segment of the second chord is 2 cm. Find the other segment of the second; measure the radius of the circle and also obtain it by calculation.

66. In a fixed straight line OP a length OA of 1·5″ is taken; with O as centre and radii 2·1″ and 3·5″ two concentric circles are described. If any other straight line from O cuts these circles in X and Y, prove that the circle through the points A, X, Y, meets OP again in a fixed point, and find the distance of this point from O.

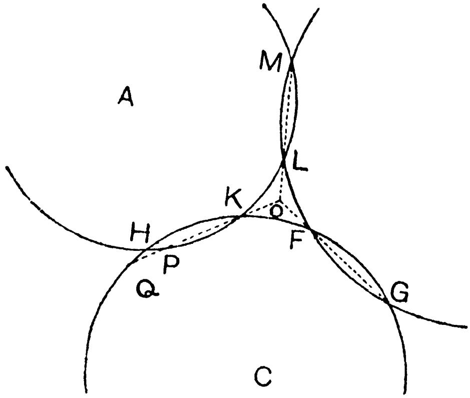
67. Three points A, B, C, are taken in a straight line so that AB = 3 cm., BC = 1 cm., and through C another straight line DCE is drawn making an angle of 60° with AC. Describe circles to pass through A and B and to touch DE; measure their radii, and find them also by calculation.

68. A column of height $21'\ 4''$ stands on a horizontal plane and is surmounted by a statue of height $5'\ 4''$: find points on the horizontal plane at which the statue subtends maximum angles.

69. AB is a finite straight line and CD an indefinite one : find the point P in CD at which AB subtends the greatest possible angle.

70. Two circles intersect, and from a point on their common chord two chords are drawn one to each circle ; prove that their four extremities are concyclic.

*71. Three circles intersect two and two : prove that their common chords are concurrent.



Solution. Let three ○s with centres at A, B, C intersect two and two in F and G, H and K, L and M: then the three common chords FG, HK, LM, are concurrent.

Let FG, LM meet in O; let OK, produced if necessary, meet the ⊙ A in P and, if possible, the ○C in Q.

From the ⊙ A, rect. OK OP = rect. OL OM, [VIII]

and from the ○ C, rect. OK.OQ = rect. OF.OG.

But rect. OL OM = rect. OF.OG, from the ⊙ B;

∴ rect. OK.OP = rect. OK OQ, so that OP = OQ.

This is impossible unless P and Q are the same, that is, unless OK goes through H.

∴ FG, LM, HK are concurrent.

72. A, B, C, D are four points in a straight line in order : find another point O in the line such that (i) rect. OA.OB = OC.OD, (ii) rect. OA.OC = rect. OB.OD.

*73. Two circles intersect: prove that the locus of points from which equal tangents can be drawn to them is a straight line. Shew how to find points from which equal tangents can be drawn to two non-intersecting circles.

74. Produce a given finite straight line AB both ways to P and Q, so that the rect. PA.AQ is equal to a given square and the rect. PB.BQ is equal to another given square.

75. The diameter AB of a circle is produced to P: draw through P a secant PQR such that the circle on QR as diameter may touch AB.

76. From a given point P outside a circle draw a secant PQR so that PQ may be equal to QR. Find the distance of P from the centre for the problem to be possible.

77. How far must any diameter of a given circle be produced so that the tangent to the circle from the end of the produced part may be of given length?

78. A′ is the midpoint of the side BC of a triangle ABC inscribed in a circle, and AA′ meets the circle in K: prove that $AB^2+AC^2=2AA'.AK$.

---

79. AB is a common tangent to two circles and P is its midpoint; the secants PQR, PST are drawn cutting the circles, and M, N are the midpoints of QR, ST respectively. Prove that $PM^2+SN^2=PN^2+RM^2$.

80. AB is the diameter and C the centre of a semicircle; the radius CD is perpendicular to AB, and a chord AQ of the semicircle meets CD in P. Prove that $AP.AQ=2CD^2$.

81. AC and AD are two tangents to a given circle; from a point E on the circle a straight line is drawn meeting AC in P and AD in Q, so as to be bisected at E. Prove that $2PC^2+2QD^2=PQ^2$.

*82. From an external point P a tangent PA and a secant PBC are drawn to a circle: prove that any circle passing through B and C cuts orthogonally the circle whose centre is P and radius PA.

83. From an external point P tangents are drawn to a given circle of centre C; the chord of contact meets PC in U. Prove that any circle passing through P and U cuts the given circle at right angles.

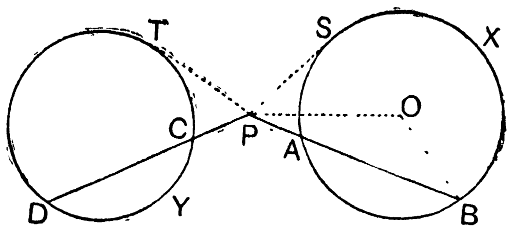
* 84. Prove the following construction for dividing a given straight line AB internally and externally in medial section.—Erect BE perpendicular to AB and $=\frac{1}{2}AB$; with E as centre and radius EB draw a circle and let AE meet it in P and Q; with A as centre and radii AP, AQ, describe circles meeting AB and BA produced in H and K respectively: then H and K are the internal and external points of medial section.

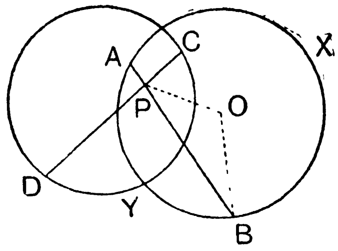
85. AB is the diameter of a semicircle, and P and Q are two points on its arc; AP and BQ meet in L, AQ and BP in M.

Prove that $\quad AQ\ QM=BQ.QL$

and $\quad AQ^2=BQ\cdot QL+PA.AL$.

§ 6. RADICAL AXIS.



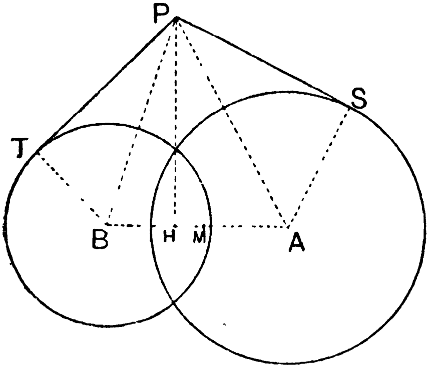


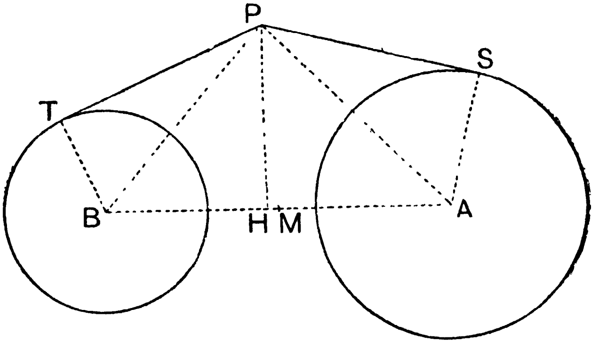
If any chord, AB, of a fixed circle ABX, whose centre is O, is drawn passing through a point P, we have proved that the rectangle of its segments is always the same, and $= OP^2 - (\text{radius})^2$ or $(\text{radius})^2 - OP^2$, according as P is outside or inside the circle. If CD is a chord of another circle also passing through P, the rectangle of the segments PC, PD is constant for it, though not of the same magnitude as PA.PB. When the two rectangles are equal it is found that P moves on a straight line. If P is outside the circles, the tangents PS, PT, to them are also equal, for $PS^2 = PA.PB$ and $PT^2 = PC.PD$. Hence we adopt the following

Def. The *radical axis* of two circles is the locus of a point from which *equal tangents* can be drawn to the circles; or, more generally, it is the locus of a point whose *powers* with respect to the two circles are *equal.*

Proposition X, Theorem.

The radical axis of two given circles is a straight line.





Let A and B be the centres of the given circles of radii a and b respectively; let P be any point whose powers with respect to the two circles (or the tangents PS, PT, from which) are equal: to prove that the locus of P is a straight line.

Join PA, PB, AB, AS, BT, bisect AB in M, and draw PH perp. to AB.

$\because$ the powers of P with respect to the two circles are equal,

$\therefore$ $PB^2 - BT^2 = PA^2 - AS^2$; [VIII, *Cor.* 3]

$\therefore$ $AS^2 - BT^2 = PA^2 - PB^2$

$= 2\,AB.HM.$ [1]

But AS, BT, AB are all fixed, and M is a fixed point;

$\therefore$ H is a fixed point in AB,

and HP, being perp. to AB at this fixed point, is a fixed straight line.

$\therefore$ all positions of P at which the powers are equal lie on this fixed straight line.

Again, the line HP being constructed by taking H at the fixed distance $(a^2 - b^2) \div 2AB$ from the midpoint M towards the smaller circle, and the perp. to AB at H being drawn, at every point of it the powers of the two circles are equal.

∴ the radical axis of the two circles is the fixed straight line HP.

Cor. 1. When two circles interesect, the radical axis is their common chord produced both ways. [See Fig. 1, and Exer. 73.]

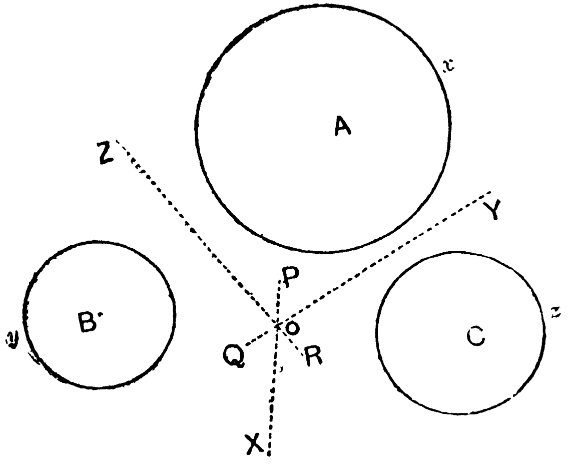
From points of the chord within the circles no tangents can be drawn; but the powers of such points are still equal [VIII].

Cor. 2. When two circles touch, the radical axis is their common tangent at the point of contact.

Cor. 3. The radical axis of two circles bisects each of their common tangents.

Proposition XI. Theorem.

The radical axes of three circles taken in pairs are either concurrent or parallel.



Let A, B, C be the centres of three circles x, y, z respectively let PX, QY, RZ be the radical axes of y and z, z and x, x and

(i) Let QY and RZ meet in O: to prove that PX passes through O.

∵ O lies on QY,
power of O for ○x = power of O for ○ z;
similarly, ∵ O lies on RZ,
power of O for ⊙x = power of O for ⊙y.
∴ power of O for ○y = power of O for ⊙ z;
∴ O lies on the radical axis of ○s y and z,
that is, PX passes through O and the radical axes are concurrent.

(ii) Let QY and RZ not meet at all; then they are parallel.

Now AC is perp. to QY and AB to RZ; [X]

∴ A, B, C are in one straight line.

∴ the third radical axis, PX, which is perp. to BC, is parl. to each of QY and RZ.

Cor. 1. When three circles intersect two and two, their three common chords are either all concurrent or all parallel. [See Exer. 71.]

Cor. 2. If a circle touches two others, the tangents at the points of contact meet on the radical axis of the two.

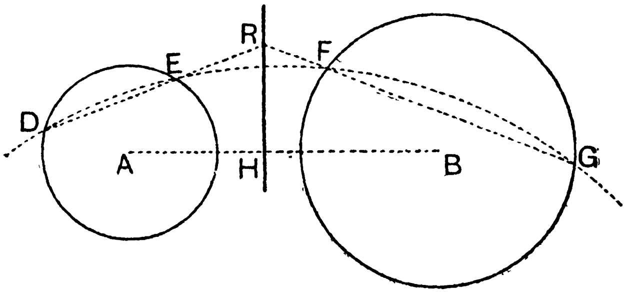
Def. The point of concurrence of the three radical axes of three circles taken in pairs is called the *radical centre* of the circles.

Proposition XII. Problem.

To construct the radical axis of two given circles.

(i) When the circles intersect or touch, the common chord or the common tangent at the point of contact is the radical axis.

(ii) When the circles, whose centres are A and B, do not touch or intersect,



describe any ○ to cut them at D, E and F, G respectively. Let DE, FG produced meet in R; draw RH perp. to AB. then RH is the radical axis.

∵ rect RD.RE = rect. RF.RG, from the ○DEFG,

∴ the powers of R for the two given ○s are equal;

∴ R lies on their radical axis.

And ∵ the radical axis is perp. to the line of centres, [X]

∴ RH is the required radical axis.

Def. When every two of three or more given circles have the same radical axis, the circles are said to be *co-axal.*

If several circles pass through the same two points, P and Q, they form a *co-axal* system; for the straight line PQ is the radical axis of every pair of circles of the system [X, *Cor.* 1]. Such circles may be called *common-point* co-axal circles.

Suppose several non-intersecting circles to be *co-axal.* The tangents drawn to them from any point of the radical axis are equal, and hence their points of contact all lie on a circle, which is orthogonal to each of the given circles. (See the *Def.* after Prop. IX.) This circle cuts the line of centres of the circles of the co-axal system in two fixed points; these may be considered as circles of the system of radius zero, and are called the *limiting points* of the co-axal circles.

Exercises.

86. Construct the radical axis of two circles of radii 2·5 and 1·5 cm., whose centres are (i) 5 cm. (ii) 3 cm. apart; and verify the position by calculating the distance of the radical axis from the midpoint of the line of centres.

87. If two circles are unequal, their radical axis is nearer the *centre* of the *smaller* but nearer the *circumference* of the *larger* circle.

*88. If with any point on the radical axis of two circles as centre a circle be described with radius equal to the tangent drawn to either circle, this circle cuts the given circles orthogonally.

*89. Three circles touch each other two and two: prove that their common tangents at the points of contact meet in a point which is the centre of a circle touching the sides of the triangle formed by the three centres.

*90. Describe a circle to cut three given circles orthogonally.

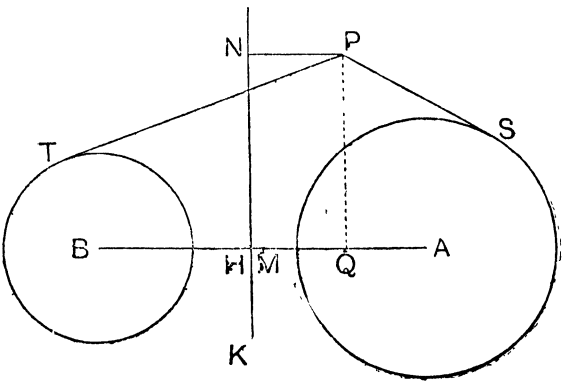
*91. With any point on the radical axis of a system of non-intersecting circles as centre and the constant length of the tangents drawn to the circles as radius a circle is described: prove that this circle cuts the line of centres in two fixed points.

92. The four tangents drawn from a point P to two given circles are equal if the chords of contact intersect in Q, prove that PQ is perpendicular to the line of centres.

93. The locus of a point the sum of whose powers with respect to two given circles is constant is a circle.

*94. The difference of the squares on the tangents from any point to two given circles is equal to twice the rectangle contained by the distance between the centres and the distance of the point from the radical axis of the circles.

Solution. Let A and B be the centres of two given circles, M the midpoint of AB, and HK the radical axis: then the difference of the powers of P for the two circles = 2AB.PN.



For, difference of powers
$= $ difference of the sqs. on tangents
$= \text{PB}^2 - \text{BT}^2 - \text{PA}^2 + \text{AS}^2 = \text{PB}^2 - \text{PA}^2 + \text{AS}^2 - \text{BT}^2$
$= 2\text{AB}.\text{MQ} + 2\text{AB}.\text{GM}$ [I and X]
$= 2\text{AB}.\text{HQ} = 2\text{AB}\ \text{PN}.$

When P is on the radical axis, $\text{PN} = 0$ and the powers are equal.

*95. The locus of a point the difference of whose powers with respect to two given circles is constant, is a straight line parallel to the radical axis of the circles.

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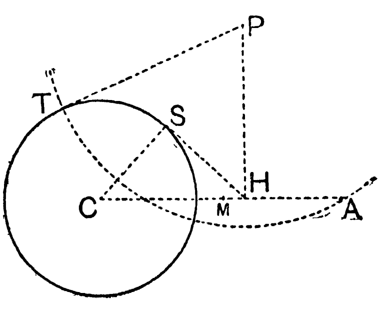
*96. If X, Y, Z are three co-axal circles, the tangents drawn from any point of Z to X and Y are in a fixed ratio.

*97. If tangents from any point P to two given circles X and Y are in a given ratio, the locus of P is a circle co-axal with X and Y.

98. The radical axes of one given circle with each of a system of co-axal circles are concurrent.

*99. Shew that the locus of a point whose distance from a fixed point is equal to the tangent drawn from it to a fixed circle, is a straight line.

Solution. Let C be the centre of the given ○ of radius a, A the given point, P one position of the moving point. Draw PH perp. to CA, and the tangent HS; join CS, and bisect AC in M.



$\because \text{PT} = \text{PA}, \therefore \text{PC}^2 - a^2 = \text{PA}^2;$
$\therefore \text{PH}^2 + \text{HC}^2 - a^2 = \text{PH}^2 + \text{HA}^2,$
$\therefore \text{HC}^2 - \text{CS}^2 = \text{HA}^2,$
i.e., $\text{HS} = \text{HA}$, so that H is also one position of the moving pt.

Again, $\because \text{CH}^2 - \text{HA}^2 = a^2,$
$\therefore$ H is a fixed pt.;

$\therefore$ the locus of P is the line perp. to CA through the fixed pt. H in it. This line may be taken as the radical axis of the circle and a circle at A of radius zero.

100. If a circle passing through a fixed point cuts orthogonally a given circle, then it also passes through another fixed point.

101. A circle cuts two given circles orthogonally: find the locus of its centre.

102. Describe a circle to cut two given circles orthogonally and (i) to pass through a given point, (ii) to have a given radius.

103. Describe a circle to cut a given circle orthogonally and to pass through two given points.

104. A circle O and three points A, B, C, are given: draw a circle through A and B such that its chord of intersection with O may pass through C.

CHAPTER III.

CONSTRUCTIONS WITH REFERENCE TO THE CIRCLE.

§ 7. Inscription and Circumscription of Regular Polygons.

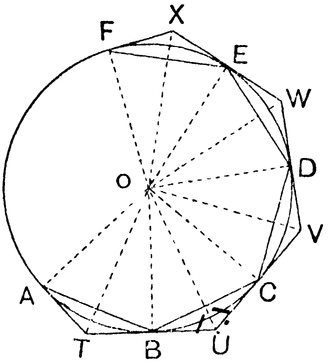
It is known that a circle can be described in or about any triangle, and that in or about a circle a triangle with given angles can be drawn ; but this does not apply to figures of a larger number of sides. Thus, taking a quadrilateral, a circle can be described about it only if the sums of its opposite angles are equal [47], and it is easily proved that a circle can be inscribed in it only if the sums of its opposite sides are equal. Similarly, rectilineal figures of more than four sides can be described in or about a circle only when they satisfy certain conditions.

In general we shall suppose the figures to be regular, that is, to have all their sides equal and all their angles equal. As the equal sides of an inscribed polygon subtend equal angles at the centre, the problem of its inscription in the circle reduces to that of dividing the whole angle (four right angles) at the centre into an equal number of parts. This may be practically done by means of the protractor : in the theoretical or strictly geometrical solution, it is understood that only the straight ruler and compasses are employed.

The angle at the centre of a circle is readily divided into four or six equal parts, by drawing two perpendicular diameters or constructing an equilateral triangle on a radius. We proceed to show how this angle may be divided into five equal parts, laying down first a general proposition about regular figures described in or about the circle.

Proposition XIII. Theorem.

If the circumference of a circle is divided into any number of equal arcs, the polygon formed by joining the successive points of division is a regular inscribed polygon, and that formed by drawing successive tangents at the points is a regular circumscribed polygon.



Let the ○*ce* of a ○ ABC be divided into n equal arcs AB, BC, CD, , and let O be the centre.

(i) Join AB, BC, CD, : to prove that the polygon ABCD... is regular.

∵ arcs AB, BC, CD, .. are equal,

∴ chords AB, BC, CD, are also equal, and the polygon is equilateral.

Again, ∵ the number of equal arcs is n, the arc AEDC contains $n-2$ of them, so does the arc BAED;

∴ arc AEDC = arc BAED, ∴ ∠ ABC = ∠ BCD, standing on equal arcs at the circumference.

Similarly, the other angles are successively equal;

∴ the polygon is equiangular.

Hence ABCD .. is the inscribed regular polygon of n sides.

(ii) Draw tangents to the circle at A,B,C,D... and let them meet successively in T,U,V,W... : to prove that the polygon TUVW... is regular.

Join OA, OB, OC, .., OT, OU, OV,... .

∵ ∠s OAT, OBT are rt. angles, ∴ ∠s ATB, AOB are supplementary; so also ∠s BUC, BOC are supplementary.

But ∠ AOB = ∠ BOC, ∴ ∠ ATB = ∠ BUC.

So also the other angles of the polygon are successively equal, and the figure is equiangular.

Again, the lines TO, UO, VO bisect the equal angles ATB, BUC, CVD [42]; hence in the Δs OTU, OUV, OU is common, $\angle$ OUT = $\angle$ OUV, and $\angle$ OTU = $\angle$ OVU, being halves of equal angles,

$\therefore$ the Δs are congruent, and TU = UV.

Similarly the other sides of the polygon are equal in succession, and the figure is equilateral.

Hence TUVW... is the circumscribed regular polygon of n sides.

Cor. If straight lines are drawn bisecting two angles of a regular polygon, the point in which they meet is the common centre of the circles inscribed in and described about the regular polygon.

Exercises.

*105. An equilateral polygon inscribed in a circle is equiangular, and consequently regular.

106. Is an equiangular inscribed polygon of a circle also equilateral?

*107. An equiangular polygon described about a circle is also equilateral. Prove this and examine the converse proposition.

108. Compare the sides of the inscribed and circumscribed equilateral triangles of any circle.

109. Draw an equilateral triangle of side 2·4 cm., and construct its inscribed and circumscribed circles. Measure their radii, and prove geometrically that one of them is double of the other.

110. A square and an equilateral triangle are inscribed in the same circle: compare their sides.

111. A square and a regular hexagon are described about the same circle: compare their areas.

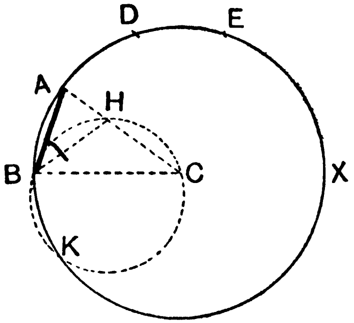
112. On a straight line 1·5″ in length draw a regular hexagon, and inscribe a circle in it. Measure the radius of the circle, and verify by calculation.

113. Prove that the area of a regular hexagon inscribed in a circle is $\frac{3}{4}$ of that of a regular hexagon described about the circle.

114. In a circle of radius r a regular octagon is inscribed: prove that the square of its side is $r^2(2-\sqrt{2})$.

Proposition XIV. Problem.

To inscribe a regular decagon in a given circle.



Let ABX be the given ○ with centre C: it is required to inscribe in it a regular decagon.

Draw any radius CA, and divide it at H so that rect. $AC.AH = CH^2$ [VII]. With centre A and radius equal to CH describe a ○ cutting the given ○ at B; join AB : then AB shall be one side of the required decagon.

Join CB, BH ; and about the △ BHC describe the ○ BHCK.

$\because AB = CH, \therefore AB^2 = CH^2 = AC.AH$; [*Cons.*]

$\therefore$ AB touches ○ BHCK [IX], and $\angle ABH = \angle HCB$. [48]

$\therefore \angle ABC = \angle ABH + \angle HBC = \angle HCB + \angle HBC = \angle BHA$. [14]

But $\angle ABC = \angle BAC$, $\therefore \angle BAH = \angle BHA$;

$\therefore BH = BA = HC$ [*Cons.*], $\therefore \angle HCB = \angle HBC$.

Now $\angle ABH = \angle HCB$, $\therefore \angle ABH + \angle HBC = 2 \angle HCB$, so that $\angle ABC = 2 \angle ACB$, and $\therefore \angle BAC = 2 \angle ACB$.

Since in the △ ABC each of the angles A and B is double of the angle C, $\therefore \angle ACB = \frac{1}{5}$ of 2 rt. angles, [15]

$\therefore \angle ACB = \frac{1}{10}$ of 4 rt. angles.

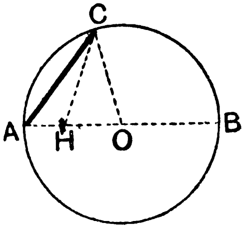
By placing at C ∠s ACD, DCE, ... each = ∠ ACB, the ○*ce* will be divided into 10 equal parts AB, AD, DE ; [38]

$\therefore$ AB is one side of the inscribed regular decagon. [XIII]

Cor. 1. By drawing tangents to the circle at the angular points A, B, D, E... of the inscribed decagon to meet successively, the circumscribed regular decagon can be constructed. [XIII]

Cor. 2. By joining the alternate angular points of the inscribed decagon, the chords of five equal arcs of the circle are obtained and the regular inscribed pentagon can be drawn. From this the circumscribed regular pentagon can be derived.

The regular pentagon can also be inscribed in a circle independently of the decagon by means of the following construction.



Through the centre O draw any diameter AOB; divide OA at H so that rect. $AO.AH = OH^2$; with centre H and radius $=$ OA describe a ⊙ to cut the given ○ at C: then the arc AC is a fifth part of the ○*ce*. For, HO is the larger segment of the radius OA when divided in medial section; it is therefore such a segment of OC and HC, each of which is equal to OA. Hence the △ OHC has each angle at the base double of the third angle, as in the Prop. above; so that the ∠ AOC is $\frac{1}{5}$ of 4 rt. angles, and AC is one side of the regular pentagon inscribed in the circle. [See also Mis. Ex. I, 20.]

If r denote the radius of a circle, the side of the inscribed regular decagon is $\frac{1}{2}(\sqrt{5}-1)r$ [Prop. VII and Note], or nearly $\cdot 618r$. If AB, BK are consecutive sides of the decagon, AK is the side of the inscribed regular pentagon, and is bisected perpendicularly at M by the diameter BCX. $\therefore$ $AB^2 = BM.BX$, so that $BM = \frac{1}{4}(3-\sqrt{5})r$. And $AM^2 = BM.MX = \frac{1}{4}(3-\sqrt{5}).\frac{1}{4}(5+\sqrt{5})r^2 = \frac{1}{16}(10-2\sqrt{5})r^2$. [See Fig. Prop. XIV.]

Thus AK, the side of the pentagon, $= 2AM = \frac{1}{2}\sqrt{(10-2\sqrt{5})}r$, or nearly 1·176 of the radius.

Exercises.

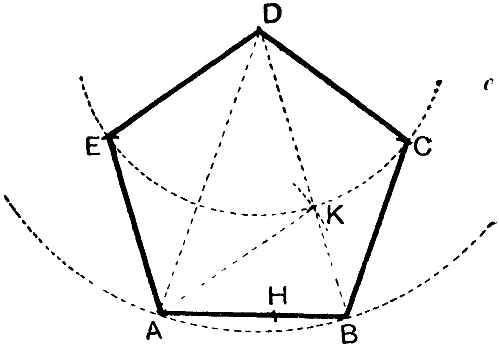
*115. Shew how to inscribe in or describe about a given circle a regular polygon of 15 sides.

116. Divide a straight line AB, 25 mm. long, at H so that $AB.HB = AH^2$. Construct triangles with sides (i) AB, AB, AH, (ii) AH, AH, BH, and measure their angles. Explain the results.

117. In a circle of radius 1·2″ inscribe a regular pentagon by ruler and compasses. Measure a side of the figure and find its area.

118. Construct an isosceles triangle each of whose angles at the base shall be (i) $\frac{1}{8}$ of the vertical angle, (ii) $4\frac{1}{2}$ times the vertical angle.

119. On a given straight line to describe (i) a regular pentagon, (ii) a regular decagon.



Solution. (i) Let AB be the given str. line. Divide it at H so that rect. AB.HB $=$ AH^2; with centres A and B and radii AB and AH respectively draw ○s to meet in K. Join AK, KB; then the Δ AKB has each of the angles at K and B double of the angle at A [Prop. XIV and Note]. Produce BK to D making KD$=$KA$=$AB; with D as centre and radius DK describe a ○ cutting the equal ○s described with A and B as centres in C and E. Then ABCDE is the required pentagon. The sides of the polygon are evidently all equal. Also, ∠ BAK$=36^0$, ∠ ABK$=$ ∠ AKB$=72^0$; hence ∠ KAD$=$ ∠ KDA $=36^0$: and ∵ the Δs AED, BCD are each congruent with Δ KDA, ∴ ∠ DAE$=$ ∠ ADE$=$ ∠ KDA$=36^0$, and ∠ BDC$=$ ∠ DBC$=$ ∠ KDA$=36^0$. Hence each angle of the polygon is 108^0.

Therefore the pentagon is regular.

(ii) To describe the regular decagon on AB, construct the Δ DAB as above. Then, ∠ KDA$=36^0$ and ∠ ABD$=72^0$, ∴ ∠ DAB$=72^0$, and DA$=$DB. With D as centre and DA or DB as radius describe a ○, and in it place chords equal to AB successively from A or B. ∵ ∠ ADB$=36^0=\frac{1}{10}$ of 4 rt. angles, ∴ each chord will be a side of the regular inscribed decagon of the ○.

120. On a straight line of length a a regular decagon is described: prove that its area is $\frac{1}{2}\ 5\sqrt{(5+2\sqrt{5})}\ a^2$.

121. In a regular pentagon prove that each diagonal is $\frac{1}{2}(\sqrt{5}+1)$ of a side; also that two diagonals which do not meet at an angle cut one another in medial section.

122. Prove the following properties of the figure of Prop. XIV:—

(i) The circle about the triangle BHC cuts the given circle in two distinct points.

(ii) If the second point of section is K, CH, HB, BK are three consecutive sides of a regular pentagon inscribed in the circle BHC.

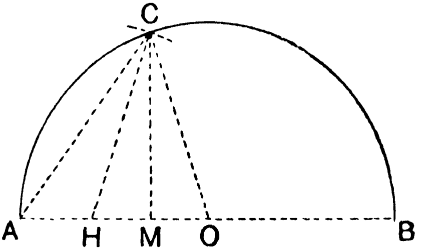
(iii) If BH meets the larger circle in Q, AQ is a side of the regular pentagon inscribed in that circle.

(iv) The circle BHC is congruent with the circumcircle of the triangle ABC.

(v) The circumcentre of the triangle ABH is the midpoint of the arc BH of the circle BHC.

*123. In any circle, prove that a chord which subtends an angle of 108° at the centre is equal to the sum of two chords subtending angles of 60° and 36° severally at the centre.

*124. Prove that the square on the side of a regular pentagon inscribed in a circle is equal to the sum of the squares on the sides of a regular decagon and a regular hexagon inscribed in the circle.



Solution. Let O be the centre and AOB a diameter of the ⊙ ; divide AO in H so that $AO.AH = HO^2$. Then HO is equal to the side of a regular inscribed decagon [XIV]; and AO, the radius, to the side of a regular inscribed hexagon. With H as centre and radius equal to OA describe a ○ meeting the given ○ in C ; join OC, CH. In the △ CHO, CH and CO are each equal to the radius, and HO is the larger segment of the radius in medial section ; ∴ ∠ CHO = ∠ COH = 2 ∠ HCO [XIV]. Hence ∠ COH = $\frac{1}{5}$ of 4rt. angles, so that by joining AC we obtain the side of a regular inscribed pentagon. Draw CM perp. to HO ; then HO is bisected in M.

Now $CA^2 = CO^2 + OA^2 - 2OA.OM$ [31]

$= AO^2 + OA^2 - OA.OH$, ∵ $OM = HM$,

$= AO^2 + OA(OA - OH) = AO^2 + AO.AH$

$= AO^2 + HO^2$: [*Cons.*]

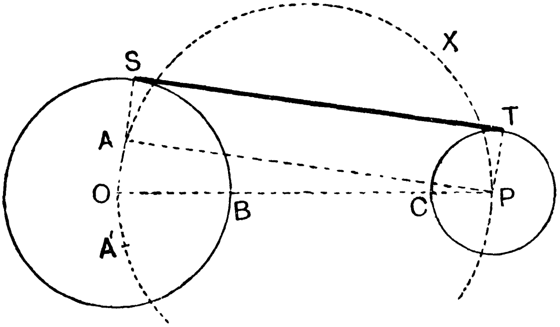
thus the result is proved. It may be verified by taking the surd values of the sides given in the Note after Prop. XIV.

§ 8. COMMON TANGENTS TO TWO CIRCLES.

Defs. When a straight line touches two or more given circles, it is said to be a *common tangent* of the circles. When two circles lie on the same side of a common tangent, the tangent is called a *direct* or *external* common tangent; and when they lie on opposite sides of it, the tangent is called a *transverse* or *internal* common tangent.

Proposition XV. Problem.

To draw a common tangent to two given circles.



Let O and P be the centres of two given ○s whose radii are a and b respectively: it is required to draw a common tangent to the ○s. On the line OP as diameter describe the ○ OXP.

(i) Place in this ○ the chord OA equal to the difference of the radii a and b; produce OA to meet the first ⊙ in S. Draw PT parl. to OS and directed the same way; join ST: then ST shall be a direct common tangent. Join AP.

∵ $OA = a - b$, and $OS = a$, ∴ $AS = b$.

∴ AS and PT are equal and parl.;

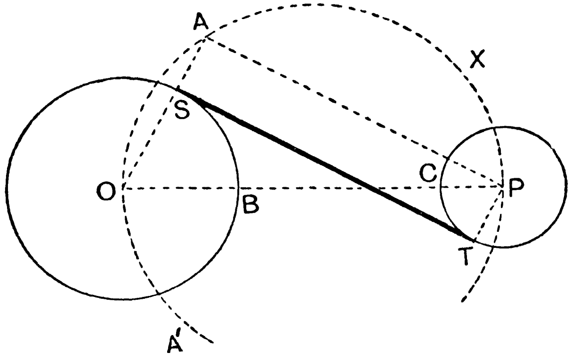
∴ ST and AP are also equal and parl. [17]

But the ∠ OAP in a semicircle is a rt. angle;

∴ ∠ OST, as also ∠ STP, is a rt. angle. [12]

Hence ST touches both the given circles. [41]

(ii) In the ○ on OP as diameter, place the chord OA equal to the sum of the radii a and b, and let OA cut the first ○ in S. Draw PT parl. to OS and directed the opposite way; join ST: then ST shall be a transverse common tangent. Join AP.



∵ OA $= a + b$ and OS $= a$, ∴ AS $= b$.
∴ AS and PT are equal and parl.;
∴ ST and AP are also equal and parl.
But ∠ OAP in a semicircle is a rt. angle;
∴ ∠ OST, as also ∠ STP, is a rt. angle.
Hence ST touches both the given circles.

By taking the arc OA′ equal to OA in the ◯ OXP, we shall get the second tangent in each of the figures above; so that in general two circles have two direct and two transverse common tangents. Let $d =$ OP, the distance between the centres. In the second case the construction is not possible if $a + b > d$, which happens when *each circle is not wholly outside the other*: in this case, therefore, there are no transverse tangents. In the first case the construction is not possible if $a - b > d$, for no chord of a circle is greater than the diameter: this happens when *one circle is wholly within the other.* In this case there are no direct tangents; and no transverse tangents too, for, of course, $a + b > d$ when $a - b > d$.

Exercises.

125. Find when two given circles have 4, 3, 2, common tangents, and when they have no common tangent.

126. Draw and measure the lengths of the common tangents of two circles for which (i) $a = 8{\cdot}4''$, $b = 3{\cdot}6''$, $d = 10''$; (ii) $a = 25$ mm., $b = 35$ mm., $d = 68$ mm.

127. Draw two circles of diameters 1·8 and 2·4 to touch externally. Construct their exterior common tangents; measure the lengths and verify by calculation.

128. Two points are at a distance of 2″ from one another: draw straight lines the perpendiculars on which from the points may be of lengths ·8″, ·6″, severally.

*129. Given the radii of two circles to be a and b, and the distance of their centres apart to be d: find the lengths of their common tangents. Discuss the formulæ, and find when there are 4, 3 and 2 solutions.

130. Two circles are given: shew how to draw a straight line cutting one of them in P and Q, and the other in R and S, so that PQ, RS may be severally equal to two given finite straight lines.

§ 9. CONSTRUCTION OF CIRCLES FROM GIVEN CONDITIONS.

We know that one circle only can be drawn passing through three points [37]: this is the simplest case of drawing a circle from three data concerning it. Instead of passing through a given point, a circle may be required to touch a given straight line or circle, to have its centre lying on a given straight line or circle, or to satisfy some similar condition. The student will find that when *three* such *independent* conditions are given, the circle may be constructed in one definite position or in a strictly limited number of positions. Thus, to touch three given straight lines intersecting two and two, four definite circles can be drawn, the inscribed and escribed circles of the triangle formed by the lines. [Prop XVIII.] If two of the lines are parallel, there will be two solutions of the problem; when all three are parallel there is no solution. In other problems too, when the data are inconsistent with each other, no circle may exist satisfying all of them.

We shall solve a few problems in which three data are taken by combining from the following classes :— (i) 1, 2, or 3 points on the circumference, (ii) 1, 2, or 3 tangent lines; (iii) 1, 2, or 3 touching circles; (iv) position of a centre-line (diameter); (v) magnitude of the radius. It will sometimes happen that the centre of the required circle moves on certain straight lines or circles, on account of the circle satisfying two of the data. In such case if a second locus of the centre could be found, the position of the centre would be determined as being at the point or points of intersection of the two loci. In this connection it is useful to remember the following simple properties of the circle:—

A circle passing through a fixed point and having a given centre-line, passes also through a second fixed point. [See 36.]

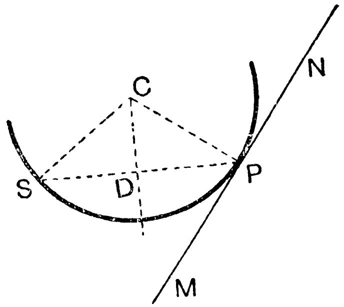
A circle passing through two fixed points has a fixed centre-line, *viz.* the perpendicular bisector of the line joining the points. [34.]

A circle touching two fixed meeting straight lines has one of two fixed centre-lines, *viz.* the bisectors of the angles made by the lines. [35.]

A circle having a given radius and touching a fixed straight line, has its centre on one of two parallel straight lines. [41.]

A circle having a given radius and touching a fixed circle, has its centre on one of two concentric circles. [43.]

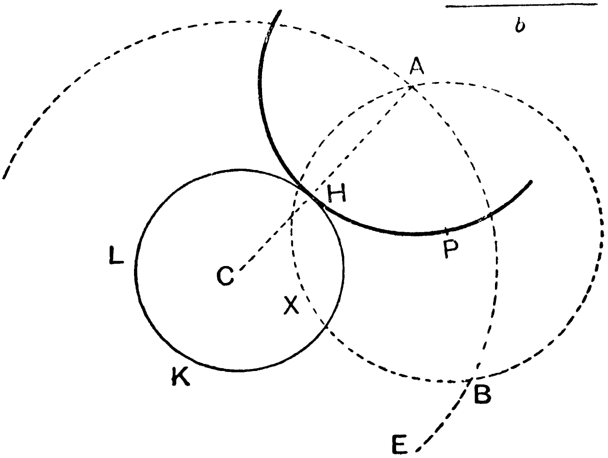
Problem 1. Draw a circle to touch a given straight line at a given point in it and also to pass through another given point.



Let MN be the given straight line, P the given point in it, S another given point. Join SP, draw PC perp. to MN, DC the perp. bisector of SP, and let them meet in C.

Then the ○ with centre C and radius CP is the required ○ for it goes through S as CS = CP, and touches MN as the ∠s at P are rt angles.

Problem 2. Draw a circle of given radius to pass through a given point and to touch a given circle.



Let b be the length of the given radius, C the centre and a the radius of the given ○HKL, and P the given point.

With P as centre and radius describe the ○ ABX; with C as centre and radius $b+a$ describe the ○ ABE, meeting the ○ ABX. Then A and B are positions of the centre of the required ○.

Join AC and let it meet the given ○ in H.

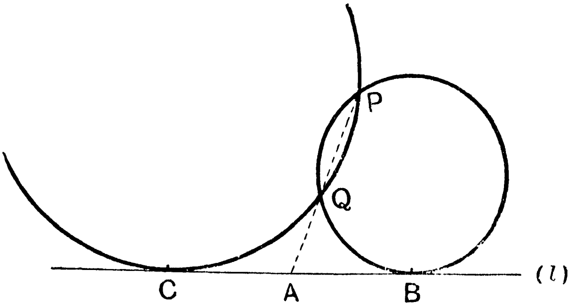
∵ $AH = AC - CH = (b+a) - a = b = AP$, ∴ the ○ with centre and radius b goes through P and H; also, as ACH is a straight line, it touches the given ○. [43]

The contact is external. So also B gives a second solution.

If with C as centre and radius $b-a$ a circle is described meeting the ○ ABX in X and Y, these points are centres of ○s touching the given ○ internally, passing through P and having the given radius.

It is seen that there are four solutions in the most general case.

Problem 3. Draw a circle through two given points so as to touch a given straight line.

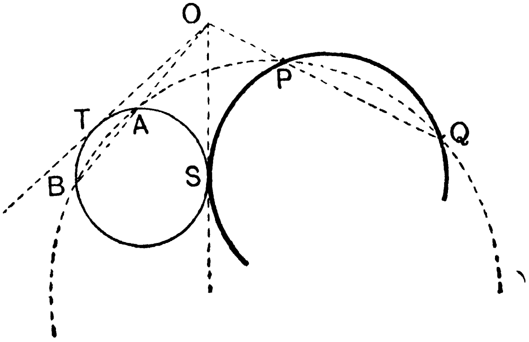


Let P, Q be the given pts., (*l*) the given line. Join PQ, and let it meet (*l*) in A; take along the line pts. B and C such that $AB^2 = AC^2 = AP\ AQ$ [V]. Then the ○ through P, Q, B, or through P, Q, C, is the required ○.

∵ $AB^2 = AP.AQ$ ∴ AB is a tangent of the ○ PQB.

So also the ○PQC satisfies the given conditions.

Problem 4. Draw a circle through two given points so as to touch a given circle.

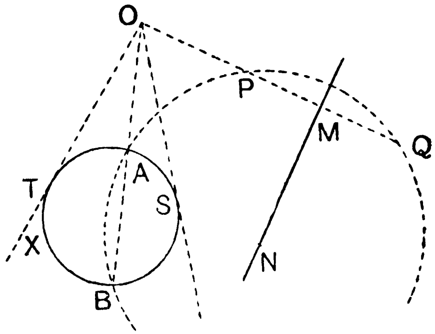


Let P,Q be the given pts., ATBS the given ⊙. Through P and Q draw any convenient ○ to cut the given ○ in A and B; let AB and PQ meet in O, and draw OT, OS to touch the given ⊙. Then the ⊙ through P, Q, S, or through P, Q, T, is the required ⊙.

∵ $OA.OB = OP.OQ$, ∴ $OS^2 = OP.OQ$, $OT^2 = OP.OQ$, [VIII and IX]

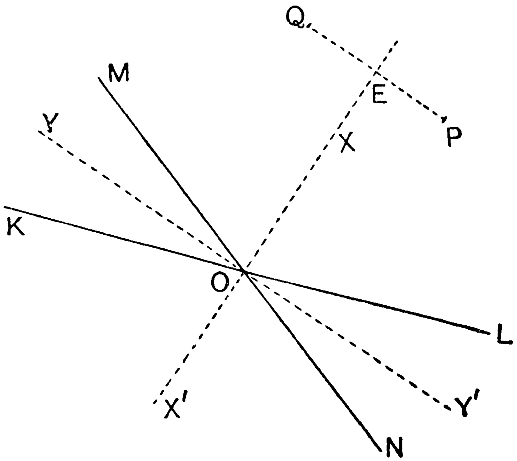
∴ OS and OT are also tangents of the ○s PQS and PQT respectively. These ⊙s, therefore, touch the given ⊙ and pass through P and Q.

Problem 5. Draw a circle through a given point so as to touch a given circle and to have its centre in a given straight line.



Let P be the given pt., ABX the given ⊙, and MN the given line. Draw PM perp. to MN, and produce it to Q, making MQ = PM. Any ○ whose centre is on MN and which goes through P, goes also through Q. The problem is now equivalent to that of drawing a ○ through P and Q so as to touch the ○ ABX.

Problem 6. Draw a circle through a given point so as to touch two given straight lines.

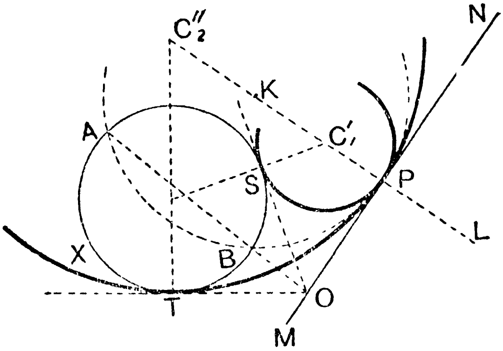


Let KL, MN be the given lines meeting at O, and P the given pt.

Draw XX′, YY′, the bisectors of the two pairs of vertically opposite angles made by the lines. Draw PEQ perp. to one of the bisectors, XX′, and make EQ = PE.

Any ○ touching KL and MN has its centre on XX′ or YY′ [35]. When the centre is on XX′, as the ○ goes through P it also goes through Q. The problem is now equivalent to that of drawing a ⊙ through P, Q, so as to touch either KL or MN. There will be only two solutions altogether.

Problem 7. Draw a circle so as to touch a given circle and also a given straight line at a given point.

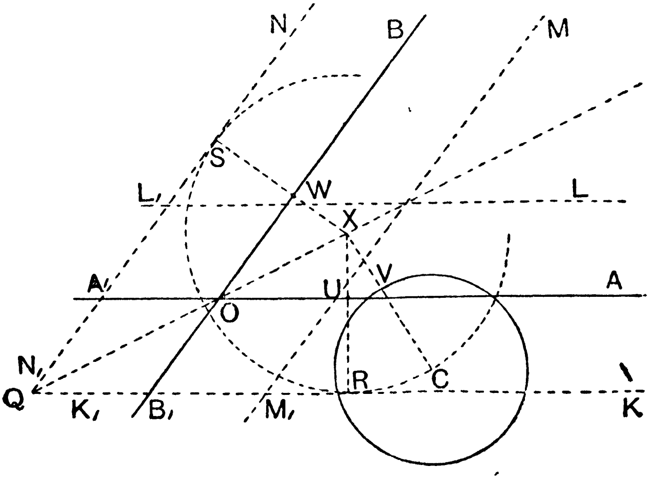


Let ABX be the given ○, and P the given pt. in the given line MN.

Draw KPL perp. to MN, and with a convenient centre K and radius KP draw a ○ to intersect the given ○ in A and B. Let AB meet MN in O, and draw the tangents OT, OS to the given ○ Then the ○ touching MN at P and passing through S or T [*Prob. 1.*] satisfies the given conditions

The points S and T of the construction above may also be thus obtained.—From the centre of the ○ ABX draw a perp to MN, meeting the ○ in Q, R, of which Q is more remote from MN. Join QP meeting the ○ in S, and RP meeting it in T. The proof is readily obtained by drawing PK and joining P and the centre of the given ○ to S and T. A similar method has to be employed to solve the more general problem—to draw a circle so as to touch a given circle and a given straight line and to pass through *any* given point.

Problem 8. Draw a circle so as to touch a given circle and two given straight lines.



Let C be the centre and a the radius of the given ○ (P), and $AOA_{,}$, $BOB_{,}$ the two given str. lines. Draw the str. lines $KK_{,}$, $LL_{,}$, at distances a from the line $AA_{,}$; and $MM_{,}$, $NN_{,}$, at distances a from $BB_{,}$. Draw a ○ to pass through C and to touch $KK_{,}$, $NN_{,}$ [*Prob.* 6]; and let X be the centre of this ⊙. Draw XR, XS, perp. to $KK_{,}$, $NN_{,}$, meeting the given lines in U, W; and join XC, meeting the given ○ in V.

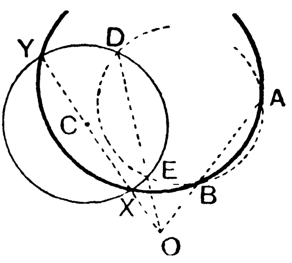
∵ $XR = XS = XC$, and $UR = WS = VC = a$,

∴ $XU = XW = XV$.

∴ the ○ described with centre X and radius XV touches the given straight lines [41] and the given ○ [43].

It should be noted that Q, the point of intersection of $KK_{,}$, $NN_{,}$, lies on the bisector of the ∠ AOB; also that there are two solutions in this case, both ○s touching (P) externally. Similarly, there will be two other ○s satisfying the data, with their centres on the bisector of the ∠ $AOB_{,}$. In the most general case when both $AOA_{,}$, $BOB_{,}$ cross the given circle, there may be eight solutions.

Problem 9. Draw a circle through two gven points so as to bisect the circumference of a given circle.



Let C be the centre of the given ○, DEX, and A, B, the given pts. Draw any convenient ○ through A, B, to meet the given ○ in D, E Produce AB and DE to meet in O, draw OXCY.

Now, rect OX OY = OE OD = OB OA,

∴ the four pts. X, Y, A, B, are cyclic [VIII, *Cor.* 2]. Draw the ○ through these, then it goes through A and B, and cuts the given ○ at the extremities of the diameter XCY

Exercises.

131. Construct a triangle, having given its base, the vertical angle, and (i) the sum or difference of the sides ;(ii) the sum or difference of the squares of the sides.

132. Find a point within a triangle at which the sides subtend equal angles. When is the solution not possible ?

133. Given the base, the height and the radius of the circumcircle of a triangle, construct the triangle.

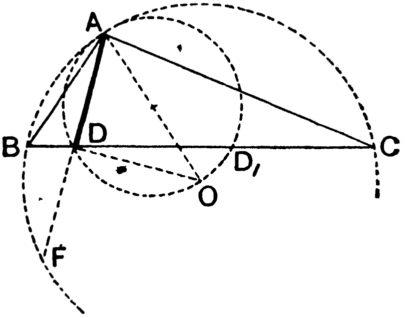
134. Given the base, the vertical angle and the radius of the incircle of a triangle, construct the triangle.

*135. A and B are two given points on the same side of a given straight line CD : determine the points on CD at which AB subtends greater angles than at other points of it.

136. A and B are two given points in a chord of a given circle : determine the points in each of the arcs cut off by the chord at which AB subtends the greatest angle.

137. From the obtuse angle A of a triangle ABC draw a straight line meeting the base in D so that rect. $BD.DC = AD^2$.

Solution. Supposing AD to be drawn, let O be the centre of the circumcircle of the Δ ABC ; produce AD to meet the ⊙ in F, and join OD.

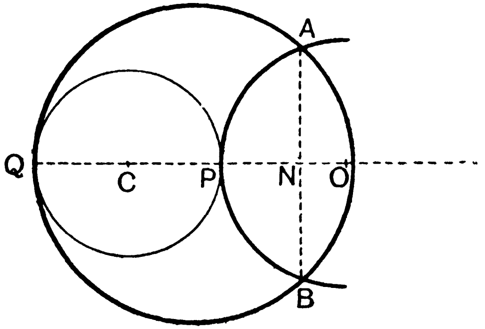


As $BD.DC = AD^2$, and also $= AD.DF$ [VIII.], ∴ $AD^2 = AD.DF$, and ∴ $AD = DF$. Hence ∠ ODA is a rt. angle [36], so that D lies on the ⊙ described on OA as diameter. Thus the problem is solved ; there are two solutions, as the ○ on OA meets BC in two pts. when the ∠ BAC is obtuse.

138. Any tangent to a circle of 3 cm. diameter is drawn : describe circles of 2·4 cm. radius to touch both the circle and the tangent. How many solutions are there ?

139. Two points are distant 1″ apart on a straight line whose distance from a given parallel line is ·8″ : construct a circle to pass through the two points and to touch the given parallel line. Measure the radius, and verify its length by calculation.

140. Two points 3 cm. apart are distant 2·7 cm. each from the centre of a circle of radius 1·2 cm. Construct circles to pass through the points and to touch the circle, and shew how to calculate the lengths of their radii.



Solution. C is the centre of the given ○, A and B are the pts.; $CP = 1·2$, $CA = CB = 2·7$, $AB = 3$.

Draw QCPN perp. to AB, which will be bisected at N. The centre of the ○ required must be in CN ; hence it touches the given ○ at P or at Q, and two solutions occur, *viz.* the ○s through A, B, P, and A, B, Q. If O is the centre of the ⊙ ABP, it is found on measurement that $OA = OP = OB = 1·6$ cm.

Let $OA = x$. Now $CN^2 = CA^2 - AN^2 = 2·7^2 - 1·5^2 = 5·04$; hence $CN = 2·25$ very nearly, so that $PN = 1·05$.

∴ $ON = x - 1·05$, and x is found from $OA^2 = x^2 = ON^2 + NA^2 = (x - 1·05)^2 + 1·5^2$. This gives $x = 1·596$. Similarly the radius of the other ⊙ is found to be 2·05 cm.

*141. Draw a circle to pass through a given point, to touch a given straight line and to have its centre on another given straight line.

142. A circle of radius a is given with its centre at the origin of co-ordinates: describe circles to touch the axes and the given circle. How many solutions are there? Prove that the radii of the circles are $a(\sqrt{2}\pm 1)$.

143. O is the centre of a circle of radius 3 cm.; a radius OA is produced to P so that AP = 4 cm. Describe circles to touch the given circle and the line OP at P, and obtain their radii by calculation.

144. Two straight lines OA, OB, include an angle of 60°; within the angle BOA a point P is taken at a distance of 1·5″ from OA and ·5″ from OB. Construct circles to touch the two given lines and to pass through P, and measure their radii.

145. A circle of radius r is drawn touching two rectangular axes: construct circles to touch the given circle and both axes, and calculate their radii.

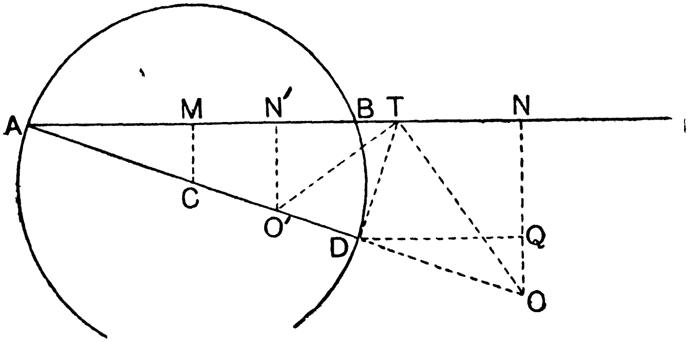
146. Circles are drawn touching two rectangular axes and going through the point whose co-ordinates are $\frac{1}{2}$, $\frac{3}{2}$: measure their radii and verify the results by calculation.

147. Circles are drawn touching the axis of x and going through the points (2, 4), (4, 2): find their radii, and prove that they touch the axis of y.

148. The sides of a triangle are of lengths 3, 4, 5: describe a circle with its centre on 4 to touch both 3 and 5. Measure the radius and verify.

149. P is a point within an angle XOY of 45°, such that the distance from OX is 6 and that from OY 4: draw circles with centres on OX to touch OY and go through P, and measure their radii.

150. Draw a circle of radius 9 and place in it a chord AB at a distance 3 from the centre C; join AC, and let it meet the circle again in D. Describe two circles to touch the given circle at D and the indefinite straight line AB; measure their radii and verify by calculation.



Solution. *Scale*, unit = ·1 inch. Draw the tangent at D meeting AB produced in T; take TN = TD. Let NO perp. to AN meet AD produced in O. Then the ○ with centre O and radius ON touches AB; it goes through D, as OD = ON from the congruent △s ODT, ONT; and it touches the given ○.

On measuring, ON is found to be ·9″, *i. e.* 9 units. Another ○ with centre O′ is obtained by taking TN′ = TD in the opposite direction; the radius is found to be 4·5 units.

Draw DQ parl. to AN. If AD = a, BD = b, OD = ON = x, then AB = $\sqrt{(a^2-b^2)}$, BT = $b^2 \div \sqrt{(a^2-b^2)}$, DT = TN = $ab \div \sqrt{(a^2-b^2)}$.

Hence, from $\mathrm{DO}^2-\mathrm{OQ}^2=\mathrm{DQ}^2$ we obtain

$$x^2-(x-b)^2=(\mathrm{BT}+\mathrm{TN})^2=\frac{b^2(a+b)^2}{a^2-b^2}, \text{ which gives } 2bx=\frac{2ab^2}{a-b};$$

$$\therefore\ x=\frac{ab}{a-b}=\frac{18\times 6}{12}=9 \text{ units.}$$

So also O′N′ may be calculated.

151. A circle with centre C and radius 2, and a straight line PQ at a distance of 4·4 from C, are given; the diameter ACB of the circle is drawn which meets the line in P at an angle of 60°. Construct circles (i) to touch the circle and the straight line at P, (ii) to touch the line and the circle at A or B. Explain the reasons for the construction in each case.

152. The sides of a triangle are of lengths 8, 9, 10; describe circles with centres on 10 to touch the other sides.

153. Draw a circle of radius 2 cm. and a chord of it distant 1·2 cm. from the centre: construct circles to touch the given circle and chord and to pass through the centre of the circle, and prove the construction.

154. Describe a circle to touch two given circles and also to touch one of their common tangents at its point of contact with one of the circles.

155. Describe a circle to pass through a given point and to touch each of two given equal circles.

156. ABC is an equilateral triangle of side 2·5 cm., and on AB as diameter a circle is described. Draw two circles to touch this circle, to pass through C, and to have their centres on AC; and prove that the radius of the smaller circle is ·5 cm.

Miscellaneous Examples II.

1. In a circle of radius 1·6 cm. inscribe a quadrilateral ABCD having the side AB 2 cm. long and the adjacent angles of 90° and 60°. Reduce the quadrilateral to a triangle on AB as base, and measure the height of the triangle above AB.

• 2. On a base 1·4″ long describe a triangle of vertical angle 40°, having (i) the sum of the other sides = 3″, (ii) the difference of the other sides = ·6″.

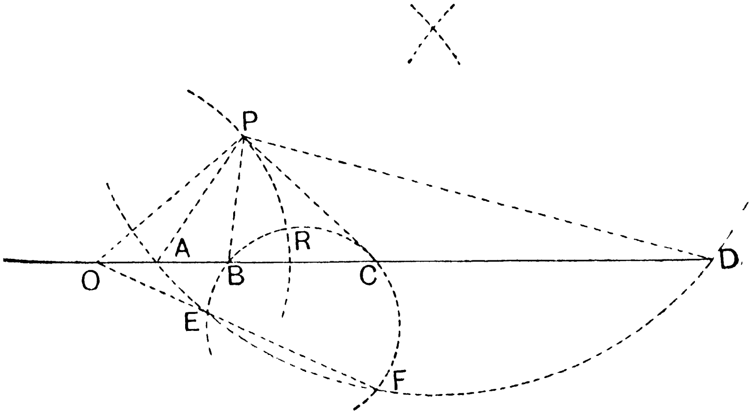
3. Construct a triangle of base 13 and vertical angle 53°, having (i) the height = 5·7, (ii) one side = 7·4.

4. Draw two circles of radii 3 and 4 with centres at a distance 8 apart: find a point from which tangents of lengths 2·65 and 3 respectively can be drawn to the circles.

5. Two circles of diameters 14″ and 4″ cut at right angles: prove that the length of their common chord is $13\frac{1}{2}\frac{1}{5}$ in.

6. Any point D is taken on the side BC of a triangle ABC; segments of circles are described on BD, DC on the side remote from A, to contain angles equal to ABC, ACB respectively. Prove that these segments intersect on the circumcircle of the triangle.

*7. A, B, C, D are four points in a straight line; shew how to find a point O in the line such that rect. OA.OD = rect. OB.OC. If each rect. = OR², prove that at any point on the circle whose centre is O and radius OR the finite lines AB, CD subtend equal angles.



Solution. Draw any conveniently large ◯s through A and D, B and C, to intersect in E and F. Let EF meet the str. line in O.

∵ OA.OD = OE.OF = OB.OC, ∴ O is the required pt.

Take OR² = OB OC [V], and let P be any pt. on the ◯ whose centre is O and radius OR.

∵ OP² = OB.OC, ∴ OP touches the circumcircle of the △PBC [IX]; ∴ ∠OPB = ∠OCP [48].

Similarly, ∠OPA = ∠ODP; ∴ by subtraction, ∠APB = ∠CPD.

8. Describe a tringle, having given the vertical angle and the lengths of the segments of the base made by the line bisecting the vertical angle.

9. Construct a square, given (i) the sum, (ii) the difference of the diagonal and side.

10. Construct a rectangle, given the perimeter and the length of either diagonal.

11. Construct a parallelogram, given one side, one diagonal, and (i) the angle between the diagonals, (ii) the angle between the sides.

12. In a triangle ABC, AL, BM, CN are the medians and AD, BE, CF the altitudes. Construct the triangle from the following data:—

(i) BC, AL, $\angle$ ABC; (ii) AD, BE, $\angle$ ABC; (iii) AL, AD, $\angle$ ABC.

13. In a right-angled triangle having an acute angle of 18°, the smallest side is to the hypotenuse as $\sqrt{5}-1$ is to 4.

14. A square of side 24 mm. is turned into a regular octagon by cutting out equal triangles at the four corners: prove that the side of the octagon is very nearly 1 cm.

15. Construct a regular octagon on a given base.

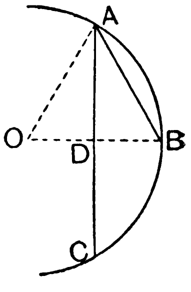
*16. If s is one side of a regular polygon of n sides, and t one side of a regular polygon of $2n$ sides, both inscribed in a circle whose radius is r, prove that $t^2 = 2r^2 - r\sqrt{(4r^2 - s^2)}$.

Solution. Let AC $(=s)$ be one side of the first polygon; bisect the arc AC in B, then the str. line AB or BC $(=t)$ is a side of the second. It is easily seen that OB bisects AC at right angles. Hence $t^2 = \mathrm{AB}^2 = \mathrm{AO}^2 + \mathrm{OB}^2 - 2\mathrm{OB}.\mathrm{OD}$ [31]

$$= 2r^2 - 2r\sqrt{(\mathrm{OA}^2 - \mathrm{AD}^2)}$$

$$= 2r^2 - 2r\sqrt{(r^2 - \tfrac{1}{4}s^2)}$$

$$= 2r^2 - r\sqrt{(4r^2 - s^2)}.$$



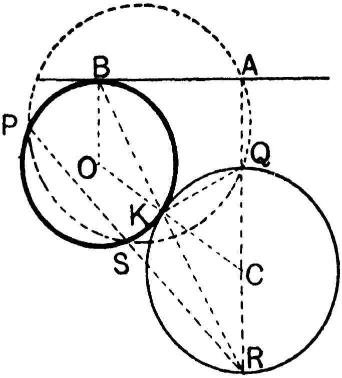
17. By means of the result above prove that the sides of the regular inscribed polygons of 3, 12, 24 sides of a circle of unit radius, are respectively $\sqrt{3}$, $\sqrt{(2-\sqrt{3})}$, $\sqrt{\{2-\sqrt{(2+\sqrt{3})}\}}$.

*18. If s is one side of a regular inscribed polygon and u one side of a regular circumscribed polygon of the same number of sides, of a circle whose radius is r, prove that $u\sqrt{(4r^2 - s^2)} = 2rs$.

19. By means of the result above find the sides of the regular circumscribed octagon and decagon of a given circle.

20. Calculate the perimeter of a regular polygon of 48 sides (i) inscribed in, (ii) described about a circle of 1″ radius.

*21. Draw a circle to pass through a given point and to touch both a given straight line and a given circle.



Solution. Let P be the given pt., AB the given str. line and C the centre of the given ⊙. Draw RCQ the diameter of the ○ perp. to AB, meeting it in A; through A, Q, P draw a ○, and let RP cut it in S. Draw the ○PSKB to go through P and S and to touch AB at B [§ 9, 3]: this is the required ○. Join RB meeting this ○ in K, and find its centre O.

∵ RK.RB = RS.RP = RQ.RA,

∴ pts. A, B, K, Q are concyclic;

∴ ∠BKQ + ∠BAQ = 2 rt. ∠s. But ∠BAQ = rt. angle, ∴ ∠BKQ = rt. angle; hence, ∠RKQ being rt., K lies on the given ○.

Again OB and RA are parl., being perp. to AB;

∴ ∠OKB = ∠OBK = ∠KRC = ∠CKR.

∴ OKC is a straight line, and ∴ the ○ PSK touches the given ○ [43].

22. Draw a circle to touch a given straight line and also to touch each of two given circles.

23. Prove that the locus of the centre of a circle which bisects the circumferences of two given circles is a straight line parallel to the radical axis of the given circles.

24. Draw a circle to bisect the circumferences of three given circles.

25. In a given circle inscribe a triangle whose sides shall be parallel to those of a given triangle.

CHAPTER IV.

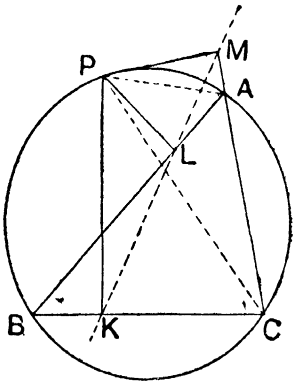
PROPERTIES OF TRIANGLES AND CIRCLES.

§ 10. THE CIRCUMCIRCLES AND THE INSCRIBED AND ESCRIBED CIRCLES OF A TRIANGLE.

It is known that only one circle can be described about a triangle ABC [37]. As its diameters perpendicular to BC, CA, AB, bisect these lines respectively, we conclude that *the three perpendicular bisectors of the sides of a triangle are concurrent*, the point of concurrence being the centre of the circumcircle (briefly, the *circumcentre*) of the triangle. The following proposition gives an important property of this circle.

Proposition XVI. Theorem.

The feet of the perpendiculars drawn from any point on the circumcircle of a triangle to the sides of the triangle, are collinear.



Let P be any pt. on the circumcircle of a △ ABC; draw PK, PL, PM perp. to BC, BA, AC respectively to prove that K, L, M are collinear pts.

Join ML, MK, PA, PC.

∵ ∠PMA + ∠PLA = 2 rt. angles, ∴ PMAL is cyclic [47], and ∠PML = ∠PAB [45].

So also PMCK is cyclic, and ∠PMK = ∠PCB.

But ∠PAB = ∠PCB [45], ∴ ∠PML = ∠PMK,

that is, the pts. K, L, M are collinear.

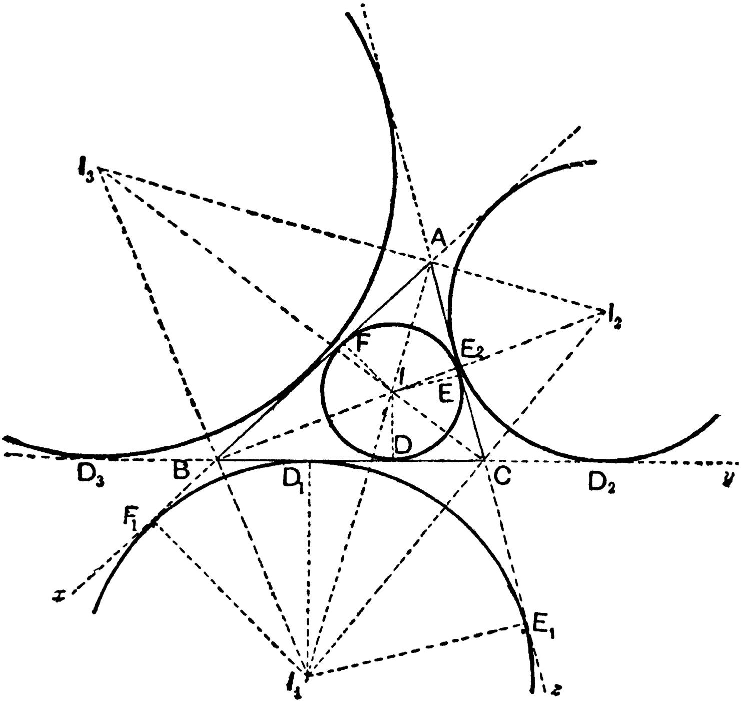
Def. The straight line KLM is called the *pedal line* of the point P with regard to the triangle ABC. It is frequently but incorrectly termed the *Simson* line of P.

The converse of the proposition may be readily proved, *viz.*, if P is a point the feet of the perpendiculars from which to the sides of a triangle are collinear, the locus of P is the circumcircle of the triangle.

A circle can be easily inscribed in a given triangle; the more general problem of drawing a circle to touch three indefinite straight lines admits of four solutions.

Proposition XVII. Problem.

To describe circles to touch three indefinite straight lines forming a triangle.



Let the str. lines Ax, By, Cz form a △ ABC: it is required to draw ○s touching each of the given lines.

Bisect the ∠s ABC, ACB by the lines BI, CI; and draw ID, IE, IF perp. to BC, CA, AB respectively.

∵ the △s BDI, BFI are congruent [6], ∴ ID = IF. Similarly, ID = IE.

∴ the ○ with centre I and radius ID passes through D, E, F, and also touches Ax, By, Cz at these pts. [41]

Again, bisect the ∠s CBx, BCz by the lines BI_1, CI_1; draw I_1D_1, I_1E_1, I_1F_1 perp. to the str. lines By, Cz, Ax.

∵ the △s BD_1I_1, BF_1I_1 are congruent, ∴ $I_1D_1 = I_1F_1$; so alsc $I_1D_1 = I_1E_1$.

∴ the ○ with centre I_1 and radius I_1D_1 passes through D_1, E_1, F_1, and also touches the given lines at these pts.

This ○ falls outside the △ ABC, and it is evident that there are two other circles of this kind, having centres at the intersections of the external bisectors at C, A and A, B, and touching each of the given lines.

Defs. The circle which touches each side of a triangle and lies within the triangle, is called the *inscribed circle* or the *incircle* of the triangle; its centre and radius are respectively the *incentre* and *inradius* of the triangle. A circle which touches one side of a triangle and the other two sides produced, so as to lie without the triangle, is called an *escribed circle* or an *excircle* of the triangle; its centre and radius are respectively an *excentre* and *exradius* of the triangle.

It is usual to denote the length of the radius of the incircle by r, that of the radius of the circumcircle by R; the lengths of the radii of the escribed circles opposite to the angles A, B, C are denoted by r_1, r_2, r_3, respectively.

The following important properties of the figure of the preceeding proposition should be noticed.

1. Join AI. As IE = IF, the △s AIE, AIF are congruent, and ∠ IAE = ∠ IAF: also, IB, IC bisect the ∠s ABC, ACB. Hence, *the bisectors of the internal angles of a triangle are concurrent.* Similarly joining AI_1, we see that $\angle I_1AE_1 = \angle I_1AF_1$; that is, I_1A is the bisector of the ∠ BAC. Hence, *the bisectors of two external angles of a triangle and the bisector of the third internal angle are concurrent.*

2. As AI, AI_1 both bisect ∠ BAC, AII_1 is a straight line: so also BII_2, CII_3 are straight lines. Hence, *a vertex of a triangle, the incentre and the excentre opposite to the vertex are collinear.* As CI_1, CI_2, bisect vertically opposite equal angles, they form one straight line: so also I_2AI_3, I_3BI_1 are straight lines. Hence, *a vertex of a triangle and the two excentres adjacent to the vertex are collinear.*

3. As AI_1, I_2I_3 bisect adjacent supplementary angles, they are at right angles to each other: so also BI_2 and I_3I_1, CI_3 and I_1I_2. Hence, I_1A, I_2B, I_3C are *the perpendiculars of the* $\triangle I_1I_2I_3$ drawn from the vertices to the opposite sides; and they *meet at one point*, I, the incentre of the triangle. We shall show that this is a property of every triangle.

4. Let the lengths of BC, CA, AB, be denoted numerically by a, b, c, respectively, and let s stand for $\frac{1}{2}$ (BC + CA + AB) or the *semi-perimeter* of the $\triangle$ ABC. Then, $\because$ AF = AE [42], $\therefore$ AE = $\frac{1}{2}$(AF + AE); so BD = $\frac{1}{2}$ (BD + BF), CD = $\frac{1}{2}$(CD + CE). $\therefore$ AE + (BD + CD) = $\frac{1}{2}$ (AB + BC + CA), that is, AE + $a = s$, and AE = AF = $s - a$. So also BD = BF = $s - b$, CD = CE = $s - c$. It may be similarly shown that $\mathrm{AF}_1 = s$, $\mathrm{BD}_1 = s - c =$ CD, $\mathrm{BD}_2 = s$, $\mathrm{CE}_2 = s - a =$ AE, etc.

Exercises.

157. If the incentre and circumcentre of a triangle coincide, the triangle is equilateral.

158. With the figure of Prop. XVII prove the following results:—(i) the midpoint of BC is the midpoint of DD_1; (ii) $\mathrm{DD}_1 = c - b$, $\mathrm{D}_2\mathrm{D}_3 = c + b$; (iii) $\mathrm{EE}_1 = \mathrm{FF}_1 = a$.

*159. If S denotes the area of the triangle ABC, prove that $rs = S = r_1(s-a) = r_2(s-b) = r_3(s-c)$.

Solution. In the Fig. of Prop. XVII,

$\triangle$ ABC = $\triangle$ BIC + $\triangle$ CIA + $\triangle$ AIB. But $\triangle$ BIC = $\frac{1}{2}$ rect. BC ID = $\frac{1}{2}ar$; $\triangle$ CIA = $\frac{1}{2}br$; $\triangle$ AIB = $\frac{1}{2}cr$; $\therefore$ $S = \frac{1}{2}r(a+b+c) = rs$.

Also $\triangle$ ABC = $\triangle$ $\mathrm{AI_1B}$ + $\triangle$ $\mathrm{AI_1C}$ − $\triangle$ $\mathrm{BI_1}c$; hence, as above,

$S = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 = \frac{1}{2}r_1(c+b+a-2a) = \frac{1}{2}r_1(2s-2a) = r_1(s-a)$.

Similarly r_2, r_3 may be found.

160. Draw a triangle of sides 16, 25, 39, and measure carefully the radii of its circumcircle, incircle and excircles. Also obtain the inradius and exradii by calculation.

161. Draw a triangle with the sides BC, CA, AB respectively 3, 3·8, 4 8 cm; let the incircle touch these respectively at D, E, F, and the excircle opposite A at D′, E′, F′., Measure BD, BD′, AF′, EE′, and verify the lengths by calculation.

162. The incircle (or the excircle opposite to the angle A) of a triangle ABC touches the sides BC, CA, AB in D, E, F respectively; FD, ED meet a straight line through A parallel to BC in Q, R. Prove that AQ = AR.

163. In a triangle ABC right-angled at A prove that (i) the indiameter = the sum of the sides − the hypotenuse; (ii) the exdiameter opposite to A = the sum of the sides + the hypotenuse.

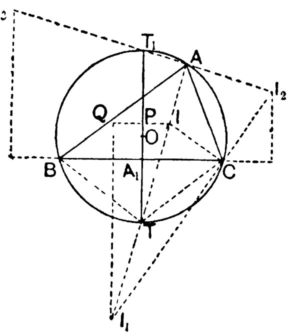
164. Two circles touch externally and each touches internally a third circle: prove that their common tangents at the points of contact meet in a point, which is an excentre of the triangle formed by the centres of the three circles.

165. If circles be circumscribed about the four triangles into which a quadrilateral is divided by its diagonals, prove that the four centres are at the vertices of a parallelogram.

166. The internal and external bisectors of the angle A of a triangle ABC meet the circumcircle respectively in E, F : prove that $EI = EI_1 = EB = EC$, and $FI_2 = FI_3 = FB = FC$.

*167. If the diameter of the circumcircle of a triangle ABC perpendicular to BC meets BC in A_1 and the circle in T, T_1, prove that $2TA_1 = r_1 - r$, $2T_1A_1 = r_2 + r_3$, and that $r_1 + r_2 + r_3 - r = 4R$.

Solution. Let O be the circumcentre. The diameter bisects BC [36], so that BA_1T, CA_1T are congruent Δs, and $BT = CT$. $\therefore$ arc $BT = \text{arc}\,CT$, and AT bisects the $\angle BAC$. Hence AT contains I, the incentre, and I_1, the excentre opposite to A. So also AT_1 bisects the external angle at A, and contains the remaining excentres, I_2, I_3. Draw IPQ, I_1Q parl. to BC, TT_1 respectively.



By joining IC, I_1C, TC, we can readily prove that $TI = TC = TI_1$ (see exercise above) ; $\therefore$ $QP = PI$, and $PT = \frac{1}{2} I_1Q$.

$\therefore \quad r_1 - r = r_1 + r - 2r = I_1Q - 2r = 2PT - 2PA_1 = 2TA_1.$

So also, as T_1 is the midpoint of I_2I_3, $r_2 + r_3 = 2T_1A_1$.

Hence, adding up, $r_1 + r_2 + r_3 - r = 2\,TT_1 = 4R$.

168. If O is the circumcentre of a triangle ABC, and A_1, B_1, C_1 the midpoints of the sides, prove that $OA_1 + OB_1 + OC_1 = R + r$.

*169. In the figure of Prop. XVI if the straight lines PK, PL, PM are revolved about P in the same direction through equal angles, prove that the points K, L, M, where they meet BC, CA, AB, are still collinear.

170. The angle made by the *pedal line* of any point on a circle with one side of an inscribed triangle, is equal to that between the lines drawn from the point to the centre of the circle and the opposite vertex of the triangle.

*171. Prove that the *Simson lines* of two diametrically opposite points of a circle, with respect to any inscribed triangle, meet at right angles.

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*172. Find a point on the circumcircle of a given triangle the pedal line of which is parallel or perpendicular to a given direction.

173. P is any point on the circumcircle of a triangle ABC, and PL parallel to BC meets the circle in L : prove that LA is perpendicular to the pedal line of P with respect to the triangle.

174. Given the base and circumcentre of a triangle, prove that the incentre moves on one of two circles.

175. Given one side of a triangle in position, the corresponding excentre and the incentre, construct the triangle.

176. Given the three excentres of a triangle, construct the triangle.

177. Given the base and vertical angle of a triangle, prove that the circumcentre is fixed, and that each excentre moves on a circle.

178. With three given points as centres describe three circles to touch each other two and two. How many solutions are there?

179. Given one angle of a triangle, its perimeter and the radius of the inscribed circle or of the escribed circle opposite to the given angle, shew how to construct the triangle.

180 The incircle of a triangle ABC touches BC at D : prove that the incircles of the triangles ABD, ACD, touch one another.

181. Find a point D in the side BC of a triangle ABC such that the incircles of the triangles ABD, ACD, may touch one another.

*182. Four straight lines in a plane form a quadrilateral: find a point in the plane such that the perpendiculars from it drawn to the lines may have their feet in a straight line.

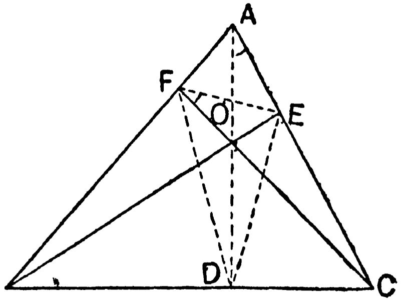
183. If perpendiculars drawn from a point P on the circumcircle of a triangle ABC to BC, CA, AB, meet the circle again at Q, R, S respectively, prove that AQ, BR, CS are each parallel to the pedal line of P.

184. Given two of the excircles of a triangle, construct the triangle.

§ 11. THE ORTHOCENTRE AND THE NINE-POINT CIRCLE OF A TRIANGLE.

Proposition XVIII. Theorem.

The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.



In the triangle ABC let the perps. BE, CF, from B, C, to the opposite sides, meet in O. Join AO: to prove that, if AO meets BC in D, AD is perp. to BC.

Join FE. ∵ each of the ∠s AFO, AEO is a rt. angle,

∴ A, F, O, E are cyclic ; [47]

∴ ∠ CFE or OFE = ∠ OAE or DAC. [45]

Again, ∵ ∠ BFC = ∠ BEC = one rt. angle, ∴ B, F, E, C are concyclic ; ∴ ∠ CBE = ∠ CFE. ∴ ∠ DAC = CBE.

Add to these the ∠ ACB;

∴ ∠ DAC + ∠ ACD = ∠ CBE + ∠ ECB,

that is, ∠ ADB = ∠ BEA = one rt. angle. [14 and *Cons.*]

Hence the three perps. meet at the point O.

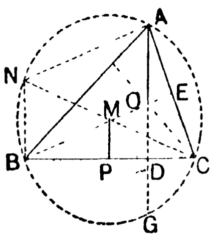
Defs. The point of concurrence of the perpendiculars from the vertices of a triangle to the opposite sides is called the *orthocentre* of the triangle. The triangle formed by joining the feet of these perpendiculars is called the *pedal* or *orthocentric* triangle of the given triangle.

The following important properties of the figure of the proposition follow as corollaries :—

1. If one of the base angles is obtuse, say the $\angle C$, the orthocentre falls outside the triangle on the same side of AE as C. In this case the above proof applies throughout, except that each of the angles DAC, CBE has to be subtracted from the $\angle ACB$. In fact, if the given triangle is AOB, it is seen that AE, BD, OF are its perpendiculars, and these concur at the point C. Similarly, B is the orthocentre of the $\triangle AOC$, and A of the $\triangle BOC$; so that *if of four points one point is the orthocentre of the triangle formed by the other three, each of the others is also the orthocentre of the triangle formed by the remaining three.* As $\angle BOC = 180 - \angle BAC$, the circles about the $\triangle$s BOC, BAC are congruent, as the angles subtended by the same chord BC of both at points of their circumferences on opposite sides of BC are equal. So also the $\triangle$s AOC, AOB have the same circumradius as the $\triangle ABC$.

2. If DF, DE are joined, it is seen that $\angle DFB = \angle C = \angle EFA$, or DF, EF are equally inclined to AB. Hence the sides of the pedal triangle are equally inclined to the sides of the given triangle; that is to say, *the sides of the given triangle bisect externally the angles of the pedal triangle.* Again as $\angle AFC = \angle BFC = 90^\circ$, therefore the remainders, the $\angle$s EFC and DFC, are equal; or, CF bisects the $\angle DFE$. Thus *the perpendiculars bisect the internal angles of the pedal triangle*, and therefore the points O, A, B, C are the incentre and excentres of the pedal triangle [XVII].

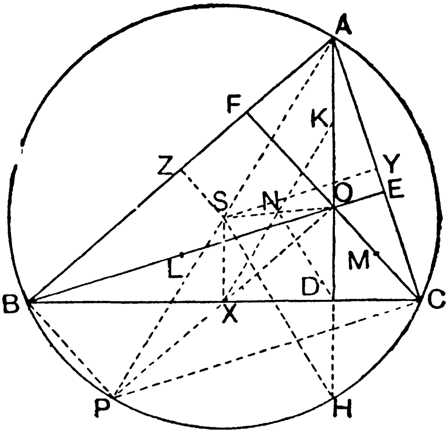
3. If the perpendicular AD meets the circumcircle ABC in G, it is seen that $\angle GBD = \angle GAC$ or DAC (from the circumcircle) $= \angle CBE$ or DBO, as proved. Hence the $\triangle$s GBD and OBD are congruent, and $DG = OD$; that is to say, *the intercept on any perpendicular between the orthocentre and the circumcircle is bisected by the corresponding side.*



4. The distances OA, OB, OC are respectively double of the distances of the circum-centre M from BC, CA, AB. Draw the diameter CMN; join NA, NB; and draw MP perp. to BC. Then $BP = PC$, also $NM = MC$: hence $NB = 2MP$. Now $\angle CBN$ = one rt. angle $= \angle CDA$, so that NB is parl. to AD; so also NA is parl. to BE, and ANBO is a parlgm. Therefore $AO = NB = 2MP$.

Proposition XIX. Theorem.

In any triangle the three midpoints of the sides, the three feet of the perpendiculars from the vertices to the sides, and the three midpoints of the lines joining the orthocentre to the vertices, all lie on a circle whose radius is half that of the circumcircle and whose centre is midway between the circumcentre and the orthocentre.



In the $\triangle$ ABC, let AD, BE, CF be the perps. meeting in O, the orthocentre; XS, YS, ZS the perp. bisectors of the sides meeting in S, the circumcentre; and let K, L, M be the midpoints of OA, OB, OC: to prove that the nine pts D, E, F, X, Y, Z, K, L, M, lie on a ○ whose centre is the midpoint of SO and radius $\frac{1}{2}$ R, where R is the radius of the circumcircle.

Bisect SO in N; join NK, SA.

$\because$ ON = NS, OK = KA; $\therefore$ NK = $\frac{1}{2}$ SA = $\frac{1}{2}$ R, and K lies on the ○ mentioned above. So also L and M lie on the same ○.

Produce OD to meet the circumcircle in H; join ND, SH.

$\because$ ON = NS and OD = DH [XVIII, 3], $\therefore$ ND = $\frac{1}{2}$ SH = $\frac{1}{2}$ R, and D lies on the same ○: so also do E and F.

Draw the diameter ASP, and join PB, PC, NX.

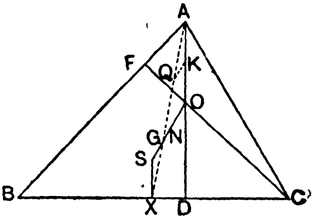
$\because$ $\angle$ PBA = 90^0 [in a semi ○] = $\angle$ CFA, $\therefore$ PB, CF are parl.; so also are PC, BE. $\therefore$ PBOC is a parlgm., and PO and BC bisect each other; hence PO goes through X the midpoint of BC.

$\because$ ON = NS, and OX = XP, $\therefore$ NX = $\frac{1}{2}$SP = $\frac{1}{2}$ R, and X lies on the same ○: so also do Y and Z.

Therefore all the nine points lie on the ○ whose centre is N and radius $\frac{1}{2}$ R.

Def. Hence this circle is called the *nine-point circle* of the given triangle.

Cor 1. It is seen that OA, OP, OH being bisected, the points of section lie on the nine-point circle; more generally, the nine-point circle is the locus of midpoints of lines drawn from the orthocentre to the circumcircle. Conversely, if the orthocentre is joined to any point on the circumcircle the joining line is bisected by the nine-point circle.



Cor. 2. Join AX, meeting SO in G; draw KQ parl. to SO to meet AX. As $AO = 2SX$ [XVIII, 4], $\therefore AK = SX$; and the Δs AQK, SGX are congruent.

$\therefore GX = AQ = \frac{1}{2} AG$, or $GX = \frac{1}{3} AX$

$\therefore$ G is the centroid of the Δ ABC.

Also $SG = QK = \frac{1}{2} GO$, or $SG = \frac{1}{3} SO$.

$\therefore$ *the centroid and nine-point centre of a triangle lie on the line joining the circumcentre and orthocentre*; the nine-point centre bisects this line and the centroid trisects it.

It can be proved more generally that if lines are drawn from the centroid to the circumcircle and produced backwards to half their lengths, the locus of their extremities (*e. g.* X in the figure) is the nine-point circle.

Exercises.

185. Given two of the excentres and the incentre of a triangle, construct the triangle.

186. The altitudes BE, CF, of a triangle ABC meet its circumcircle in G, H: prove that (i) GH is parallel to EF, (ii) A is the centre of the circle GOH, where O is the orthocentre.

*187. If S is the circumcentre of a triangle ABC, prove that SA, SB, SC are perpendicular to the sides of its pedal triangle.

188. The line joining the midpoint of BC to the orthocentre O of a triangle ABC meets the circumcircle in R : prove that ARO is a right angle.

189. The triangle ABC is right-angled at A, and a straight line *bc* perpendicular to BC meets AB in *c* and AC in *b* : find the locus of the point of intersection of B*b* and C*c*.

*190. Prove that the six lines joining the incentre I and the excentres I_1, I_2 I_3, of a triangle ABC, are bisected by the circumcircle; and that the radii of the circumcircles of the four triangles formed by these centres are each double the radius of the circumcircle of the triangle.

191. Construct a triangle of sides 2, 2·4, 3·2 cm. and carefully draw its nine-point circle. How is it situated with regard to the incircle of the triangle ?

192. If the nine-point circle of a triangle bisects one of the perpendiculars, the triangle is right-angled.

193. The base BC and the angle A of a triangle ABC are fixed, and BE, CF are its perpendiculars : prove that EF is of fixed length.

*194. Given the angles of a triangle to be α, β, γ, find those of its pedal triangle. Examine the results when the triangle is right-angled or obtuse-angled.

*195. In a triangle ABC, O is the orthocentre : prove that the circumcircles of OBC, OCA, OAB are equal, and that their centres form a triangle identically equal to ABC.

*196. With the notation of the exercise above prove that (i) $AO^2+BC^2=BO^2+CA^2=CO^2+AB^2=4R^2$; and (ii) $2(AD.AO+BE.BO+CE.CO)=a^2+b^2+c^2$, where D, E, F are the vertices of the pedal triangle.

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197. AD, BE, CF being the perpendiculars of a triangle ABC, DKL is drawn at right angles to DE to meet BE in K and BA in L : prove that $LB^2=LK.LD$.

*198. If O is the circumcentre of a triangle ABC, I, I_1, I_2, I_3 the centres of the four circles touching its sides, and R the circumradius, prove that (i) $II_1^2+I_2I_3^2=II_2^2+I_3I_1^2=II_3^2+I_1I_2^2=16R^2$, and (ii) $OI^2+OI_1^2+OI_2^2+OI_3^2=12R^2$.

Solution. It may be easily shown that $TI=TI_1=TC$. [See the solution of Exer. 167.] Hence $II_1=2TC$.

Similarly $I_2I_3=2T_1C$. $\therefore II_1^2+I_2I_3^2 = 4(TC^2+T_1C^2) = 4(TT_1^2) =16R^2$. [Compare the result (i) in Exer. 196.] Again, as T is the midpoint of II_1, $\therefore OI^2+OI_1^2=2(OT^2+TI^2)=2(R^2+TC^2)$; so also, $OI_2^2+OI_3^2=2(OT^2+T_1I_2^2)=2(R^2+T_1C^2)$. $\therefore OI^2+OI_1^2+OI_2^2+OI_3^2=2(2R^2+TT_1^2)=2(2R^2+4R^2)=12R^2$.

199. If from any point P perpendiculars PD, PE, PF are drawn to the sides BC, CA, AB of a triangle ABC, prove that $BP^2+CQ^2+AR^2=PC^2+QA^2+RB^2$. Conversely, if P, Q, R are points in BC, CA, AB satisfying the above relation, the perpendiculars to the sides at these points are concurrent.

200. Apply the result obtained above to prove that, in any triangle, (i) the bisectors of the angles, (ii) the perpendicular bisectors of the sides, (iii) the perpendiculars from the vertices on the sides, are concurrent.

201. The angular points A, B, C, of a triangle are joined to N the nine-point centre, and on AN, BN, CN produced S_1, S_2, S_3 are taken such that $NS_1 = AN$, $NS_2 = BN$, $NS_3 = CN$: prove that S_1, S_2, S_3 are the centres of the circles described about the triangles BOC, COA, AOB, where O is the orthocentre.

202. With the notation of the exercise above prove that, S being the circumcentre—

(i) $BSCS_1$, $CSAS_2$, $ASBS_3$ are rhombuses;

(ii) S is the orthocentre and O the circumcentre of the triangle $S_1S_2S_3$;

(iii) A, B, C are respectively the circumcentres of the triangles SS_2S_3, SS_3S_1, SS_1S_2; and

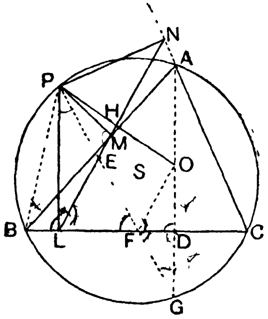
(iv) the triangles BOC, COA, AOB, S_2SS_3, S_3SS_1, S_1SS_2, ABC, $S_1S_2S_3$ all have the same nine-point circle.

*203. If P is any point on the circumcircle of a triangle and O its orthocentre, prove that the *Simson* line of P bisects OP.

Solution. PL, PM, PN are perp. to the sides, so that LMN is the pedal line of P; AD perp. to BC, contains the orthocentre O. Join OP meeting LN in H.

Let AD meet circumcircle in G, so that $DG = OD$. Join GP meeting LN in E and BC in F; also join PB, OF.

Then $\angle PLN = \angle PBA = \angle AGP$ $= \angle LPE$; and $\angle PLF$, being right, $= \angle LPF + \angle LFE$.



$\therefore$ $\angle ELF = \angle EFL$, so that $EP = EL = EF$.

Again $\angle OFD = \angle GFD$, $\because$ the Δs OFD, GFD are congruent; $\therefore \angle OFD = \angle LFE = \angle ELF$; $\therefore$ OF and LMN are parl.

Hence, as $PE = EF$, and EH is parl. to OF, $\therefore$ $PH = HO$.

*204. The Simson lines of the ends of a diameter of a circle with respect to any inscribed triangle, intersect on the nine-point circle of the triangle.

205. I is the incentre of a triangle ABC: prove that it is the orthocentre of the triangle formed by the circumcentres of BIC, CIA, AIB.

206. A′, B′, C′ are the midpoints of the sides BC, CA, AB of a triangle, and D, E, F are the feet of the respective perpendiculars: prove that the lines joining A′, B′, C′ to the midpoints of EF, FD, DE respectively, all meet in the centre of the nine-point circle.

207. Given one vertex of a triangle and two of the four points —the circumcentre, the orthocentre, the nine-point centre, the centroid: construct the triangle.

Miscellaneous Examples III.

1. Describe a circle so as to touch three given straight lines two of which are parallel. How many solutions are there?

2. In a parallelogram ABCD, BC = 2 cm., $\angle$ BCD = 60^0 : describe a circle to touch AB, BC, CD, and calculate its radius.

3. If one angle of a triangle, the radius of the incircle and the radius of one of the excircles are given, shew how to construct the triangle.

4. Given one angle of a triangle and the perimeter, construct the triangle so that the side opposite to the given angle may pass through a given point.

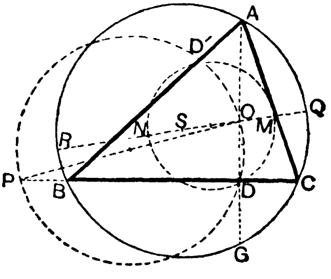
5. Two circles intersect at A and B : draw another circle touching the two circles and such that the line joining the points of contact may pass through A or B.

6. If points X, Y, Z be taken respectively in the sides BC, CA, AB of a triangle so that the quadrilaterals BZYC, CXZA, AYXB are cyclic, prove that AX, BY, CZ are the perpendiculars of the triangle.

*7. The pedal triangle is the triangle of least perimeter which can be inscribed in a given triangle.

8. The circumcircle and orthocentre of a triangle being given, describe it so that one side may pass through a given point.

Solution. Let ABC be a triangle, O its orthocentre; draw AODG to meet the cirumcircle (of centre S): then OD = DG. If QOSR is the diameter through O, and M and N the midpoints of OQ, OR, it is known that D lies on the ⊙ on MN as diameter. If BC passes through the given pt. P, $\angle$ PDO is a right angle, and D also lies on the ○ on PO as diameter. Hence the construction : draw the diameter QOSR; bisect OQ in M, OR in N; draw the ○ on NM as diameter to meet the ○ on OP as diameter in D; draw OD and PD meeting the given ○ in A (in DO produced), and in B and C.



There is a second triangle corresponding to the intersection D′ of the two circles.

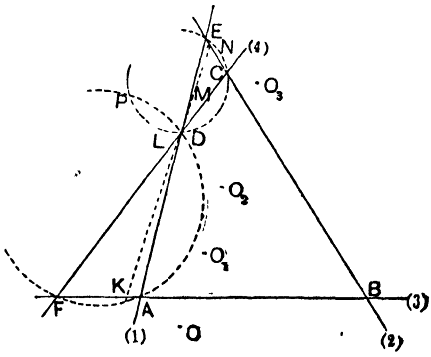
9. Given two of the angular points of a triangle and the centre of one of the circles touching its sides, construct the triangle.

*10. The circumcentre, orthocentre, incentre of a triangle ABC are S, O, I respectively: prove that the $\angle$ SAO = the difference of the angles ABC and ACB, and that AI bisects it.

11. Construct a triangle whose base and vertical angle are given and the difference of whose base-angles is known.

12. AD, BE, CF are the perpendiculars of a triangle ABC, and A_1, B_1, C_1 the midpoints of BC, CA, AB respectively: construct the triangle, when (i) D, E, A_1 are given, (ii) A_1, B_1, F are given. Can the triangle be obtained when D, E, C_1 are known?

*13. Four straight lines form four triangles when taken three and three: prove that the orthocentres of the four triangles are collinear.



Solution. Let the str. lines (1), (2), (3), (4) form the four $\triangle$s ABE, BCF, CDE, DAF; let O_1, O_2, O_3, O_4 be the respective orthocentres.

Draw ○s about the $\triangle$s CDE, ADF, meeting in P, and let K, L, M, N be the feet of the perpendiculars from P on AB, CD, AD, BC respectively. As P is on the ○CDE the pts. L, M, N are collinear [XVI]; and from the ○ADF, L, M, K are collinear. ∴ KLMN is a str. line. As LMN is the Simson line of P for the $\triangle$ CDE, and O_3 is the orthocentre, PO_3 is bisected by the line KLMN [Exer. 203]; so also PO_1, PO_2, PO_4 are bisected by the same str. line.

But K, L, M, N are collinear, ∴ O_1, O_2, O_3, O_4 are collinear.

14. If P is a point on the circumcircle of a triangle ABC, the pedal line of P cannot be in the same direction as BC, except when P is diametrically opposite to A, and then it coincides with BC.

15. Shew that the pedal line of the point in which the circumcircle is cut again by the perpendicular to BC from A, is parallel to the tangent to the circle at A.

*16. The pedal lines of three points P, Q, R, on a circle with regard to any inscribed triangle, form by their intersection a triangle equiangular to the triangle PQR.

17. Let P be a point on the circumcircle of a triangle ABC, and QR any chord of the circle parallel to the Simson line of P: prove that PR is parallel to the Simson line of Q, and PQ to the Simson line of R.

CHAPTER V.

RATIO AND PROPORTION.

§ 12. DEFINITIONS AND ELEMENTARY THEOREMS.

The *ratio* of two magnitudes of the same kind, A and B, is the number showing how often one of them A contains the other B, or what part or parts A is of B.

This ratio is written in the form A : B, and is known to be *measured by the fraction* A/B; A is called the *antecedent* of the ratio and B the *consequent.*

If A and B are geometrical magnitudes, *e. g.*, two lines or areas or angles, they may be expressed in terms of the *same unit* (unit length or area or angle) by the *numbers* a and b respectively; then the ratio A : B is conveniently measured by the ratio $a : b$ or the fraction a/b.

In many cases it will not be possible to find the fraction a/b in *finite terms.* The ratio of two lines of lengths $3\frac{1}{4}''$ and $5\frac{2}{8}''$ is that of the numbers 39 and 68 when $\frac{1}{12}''$ is the unit; but it is not possible to express the ratio of the diagonal of a square to one side by such numbers, however small the unit may be. In such cases, when no common unit can be found to measure both terms, the magnitudes are said to be *incommensurable.* Similarly, as the square root of 3 by the ordinary process does not give a terminated result, the ratio $\sqrt{3} : 1$ is incommensurable. It may, however, be expressed to any required degree of approximation; for it will be found that $\frac{\sqrt{3}}{1} > 1{\cdot}73$ and $< 1{\cdot}74$; $> 1{\cdot}732$ and $< 1{\cdot}733$; $> 1{\cdot}73205$ and $< 1{\cdot}73206$; and so on.

When the ratio of two magnitudes of the same kind, A and B, is equal to that of two other magnitudes of the same kind, C and D (but not necessarily of the same kind as A and B), the four magnitudes are said *to be proportionals* or *to form a proportion.* The equality is thus expressed,—

$$A : B = C : D \text{ or } A : B :: C : D.$$

A and D are called the *extreme* terms, B and C the *mean* terms of the proportion. Again the antecedents, A and C, of the two equal ratios are said to be *homologous* or *corresponding* terms; so also the consequents, B and D.

The antecedent and consequent of each ratio must be terms of the same kind, so that such a result as A:C = B:D cannot occur, unless C and D are of the same kind respectively as A and B. But when the numerical measures of the ratios are employed, the terms may stand in any order algebraically admissible, and we can have the numerical proportion $a:c=b:d$.

When four magnitudes are in proportion, the fourth is said to be the *fourth proportional* to the first three.

When three magnitudes are in continued proportion, so that A:B = B:C, the third (C) is said to be the *third proportional* to the first two, and the second (B) is said to be the *mean proportional* to the first and third.

The square, A^2/B^2, of a ratio, A/B, is often called the *duplicate ratio* of A : B.

When quantities are in proportion, many other proportions and inferences can be derived by the laws of Algebra : the more important of these are given below

(i) When four quantities are in proportion, the product of the extremes is equal to the product of the means.

Let a, b, c, d be the numerical measures of the quantities.

Then $a/b=c/d$, by definition ·

multiply each side by bd; $\therefore\ ad=bc$.

Cor. 1. If a, b, c, d are the measures of straight lines, it is known that ad, bc are the square units contained in rectangles with sides a and d, b and c. Hence, when four straight lines are in proportion *the rectangle contained by the extremes is equal to that contained by the means.*

Cor. 2. When three straight lines are in proportion, so that A:B = B:C, *the rectangle contained by the extremes is equal to the square on the mean.*

(ii) Conversely, if the product of one pair of quantities is equal to the product of another pair, the four quantities are in proportion so that the quantities of either pair are both extreme terms or both mean terms.

Let $ad = bc$: divide each side by bd;

$\therefore \quad \frac{a}{b} = \frac{c}{d}$ or $a:b = c:d$.

Cor. If the product of a pair of quantities is equal to the square of another quantity, the three are in continued proportion, the latter quantity being the repeated mean term.

(iii) If $a:b = c:d$, then $b:a = d:c$.

For $a/b = c/d$: divide unity by each ; $\therefore$ $b/a = d/c$.

The use of this theorem is generally indicated by the term "*invertendo*".

(iv) If $a:b = c:d$, than $a:c = b:d$ [*alternando*].

For $ad = bc$, by (i) : divide each side by cd;

$\therefore \quad \frac{a}{c} = \frac{b}{d}$, or $a:c = b:d$.

(v) If $a:b = c:d$, then $a \pm b:b = c \pm d:d$.

For $a/b = c/d$: add and subtract unity ;

$\therefore \quad \frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$, or $a \pm b:b = c \pm d:d$.

The inference is indicated by *componendo* or *dividendo*, according as the positive or negative sign is taken : in the latter case $a > b$ and $c > d$.

Cor. 1. Similarly we can infer $a:a \pm b = c:c \pm d$.

Cor. 2. By componendo, $\frac{a+b}{b} = \frac{c+d}{d}$;

and by dividendo, $\frac{a-b}{b} = \frac{c-d}{d}$:

divide the first equation by the second, term by term, and

we get $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

or $a+b:a-b = c+d:c-d$ [*componendo* and *dividendo*].

(vi) If three quantities are in continued proportion, *the ratio of the first to the third is the duplicate ratio of the first to the second* (or the second to the third).

Let $a:b = b:c$, then $ac = b^2$.

$\therefore \quad \frac{ac}{c^2}$, *i. e.*, $\frac{a}{c} = \frac{b^2}{c^2} = \frac{a^2}{b^2}$, $\because \quad \frac{a}{b} = \frac{b}{c}$.

(vii) When several ratios are equal, *each of them is equal to the ratio of the sum of the antecedents to the sum of the consequents.*

Let $\frac{a}{p}=\frac{b}{q}=\frac{c}{r}=\frac{d}{s}$;

then each of these $=\frac{a+b+c+d}{p+q+r+s}$.

For let the common value of the ratios be m;

then $\frac{a}{p}=m$, $\therefore a=pm$.

So also $b=qm$, $c=rm$, $d=sm$.

Hence $a+b+c+d=m\,(p+q+r+s)$;

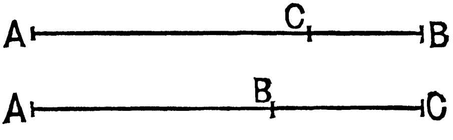
$$\therefore \quad \frac{a+b+c+d}{p+q+r+s}=m=\text{each of the given ratios.}$$

It can be similarly proved that each ratio $=\frac{a-b}{p-q}=\frac{b-c}{q-r}$, etc.

§ 13. PROPORTIONAL DIVISION OF STRAIGHT LINES.

In § 4 we have given the definition of the segments of a straight line made by a point and shown how these have to be taken with proper algebraical signs to make up the given line.

If A, B are the extremities of the line and C the point of section, the ratio of the segments is AC : CB; and the line AB is said to be divided at C in this ratio. It is seen that in the first case the ratio is positive, while in the second it is negative as CB is of opposite sign to AC.



In case (1) the ratio may be made to have any positive value by making C, the point of division, move along AB from A to B, the value rising from zero in the beginning to a number larger than any assignable quantity. In case (2) by making C move from B to infinitely distant points in AB produced, the ratio may be altered from a negatively infinite value to a value as nearly unity as possible and negative; while by taking C in BA produced, the values of AC : CB range from 0 to −1.

The following theorem gives an important result about the point of division of a line in a given ratio.

Proposition XX. Theorem.

• *A given straight line can be divided in a given ratio in only one point either internally or externally.*

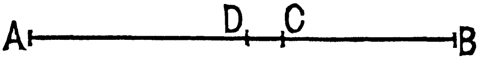


Fig. 1

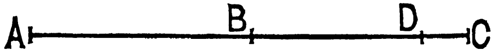


Fig. 2

Let AB be the given finite line, and let it be supposed to be divided, internally in Fig. 1 externally in Fig. 2 in a given ratio. Then there will be only one pt. of division, C, in each case. For, if possible, suppose D to be a second pt. of division in the same ratio, so that $\frac{AC}{CB} = \frac{AD}{DB}$.

$\therefore$ (1), $\frac{AC + CB}{CB} = \frac{AD + DB}{DB}$; [*Componendo*]

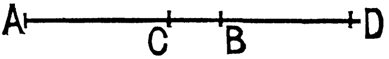
and (2), $\frac{AC - CB}{CB} = \frac{AD - DB}{DB}$. [*Dividendo*].

Hence in both cases $\frac{AB}{CB} = \frac{AB}{DB}$;

$\therefore$ CB = DB, which is impossible, unless C and D coincide.

$\therefore$ there is only one such point of division.

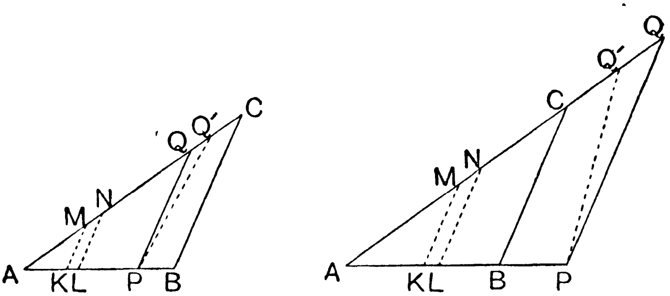
Def. When a finite straight line is divided internally and externally in the *same ratio*, it is said to be *harmonically* divided.



Thus if in the accompanying figure AC : CB equals AD : DB, the line AB is said to be harmonically divided at C and D. It should be remembered that the ratios are only *numerically* equal, as one is positive and the other negative; and the correct algebraical statement is $\frac{AC}{BC} = -\frac{AD}{DB}$.

Proposition XXI. Theorem.

If a straight line is drawn parallel to one side of a triangle, divides the other two sides, internally or externally, in the ame ratio. Conversely, if a straight line is drawn so as to ivide in order two sides of a triangle, internally or externally, n the same ratio, it is parallel to the third side.



(i) Let ABC be any Δ, and let PQ parl. to BC cut AB, AC in ,Q respectively : to prove that AP . PB = AQ : QC.

Take KL a common measure of AP and PB, so that AP contains t p times and PB, q times.

∴ AB can be divided into $p+q$ or $p-q$ parts, each = KL, and is one of the pts. of division.

Through each pt. of division draw str. lines parl. to PQ, and let M, LN be two consecutive parl. lines out of these.

Then AC is also divided into $p+q$ or $p-q$ parts, each = MN 19], and Q is one of the pts. of division.

Hence $\frac{AP}{PB}=\frac{p.KL}{q.KL}=\frac{p}{q}$, $\frac{AQ}{QC}=\frac{p.MN}{q.MN}=\frac{p}{q}$;

$\therefore \frac{AP}{PB}=\frac{AQ}{AC}$, or AP: PB = AQ : QC.

(ii) Let the sides AB, AC of a Δ ABC be divided proportionally n order, so that AP : PB = AQ : QC : to prove that PQ is parl. to BC.

If PQ is not parl. to BC, draw, if possible, PQ′ parl. to it.

Then AQ′ : Q′C = AP:PB, by the first part:

ut AQ : QC = AP : PB ; [Hyp.]

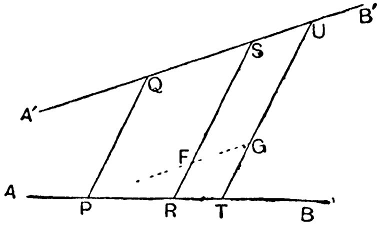
∴ AQ′:Q′C = AQ:QC,
hat is, AC is divided at Q and Q′ in the same ratio, either nternally or externally.

∴ Q and Q′ coincide, [XX]
nd hence PQ is parl. to BC.

Cor 1. We shall have also AP:AB = AQ:AC, by reasoning as in the first part of the proposition. Or this result may be obtained by componendo or dividendo. [See § 12, v, *Cor. 1.*]

Cor. 2. If any two str. lines are cut by any number of parl. str. lines, the corresponding segments are proportional. If AB, A′B′ are so cut by PQ, RS, TU, then, drawing PFG parl. to QSU, we have PR:RT = PF:FG, by the Prop.; but PF = QS, FG = SU [16].

Hence PR:RT = QS:SU.



Exercises.

208. Prove Prop. XXI in the case when P is in BA produced.

209. AD, BC are the parl. sides of a trapezium ABCD, and points P and Q are taken in AB, CD respectively so that AP : PB = DQ : QC : prove that PQ is parallel to AD or BC.

210. In a quadrilateral ABCD the diagonals intersect at O and AO : OC = DO : OB : prove that two sides of the quadrilateral are parallel.

211. Draw to scale a line 45′ long and divide it (i) internally in the ratio 2 : 3, (ii) internally as well as externally in the ratio 1 : 5. Verify the results by calculation.

212. In the two figures of Prop. XXI, AP = 9″, QC = 8″, AQ = 2PB : find AB and AC.

213. D, F and E, G are points on the sides AB and AC respectively of a triangle ABC, such that DE, FG are each parallel to BC: if AD = 4, DB = 3, AE = 6, GC = 1, find DF and EG.

214. D is any point in the side BC of a triangle ABC; a straight line parallel to BC meets AB, AD, AC (produced if necessary) in M, O, N respectively. Prove that MO : ON = BD : DC.

*215. ABC is a triangle, and any line MN parallel to BC meets AB in M and AC in N: prove that the locus of the intersection of BN and CM is a straight line.

216. The straight lines AOB, COD meet in O so as to make AO:OB = CO:OD: if P, Q are the midpoints of AB, CD respectively, prove that PQ is parallel to AC or BD.

217. Two circles touch (internally or externally) at A, and a straight line through A meets them again in P and Q severally: prove that AP:AQ equals the ratio of the radii of the circles.

218. D is any point in the side BC of a triangle ABC, and DE, DF parallel to AC, AB respectively meet AB, AC at E, F respectively: find the locus of the midpoint of EF.

219. ABCD is a parallelogram; P is a point in AD such that AP:PD = 3:8. If BP and AC meet in L, find the ratio AL:LC.

220. ABCD is a parallelogram; H and K are points in AB and CD such that $AH=\frac{1}{3}AB$, $CK=\frac{1}{3}CD$. Find the ratios of the segments of AC made by HD and KB.

221. In a triangle ABC, BD is taken along BC equal to $\frac{1}{3}$ of BC; AD is joined and bisected in F. If BF meets AC in G, find the ratio AG:AC.

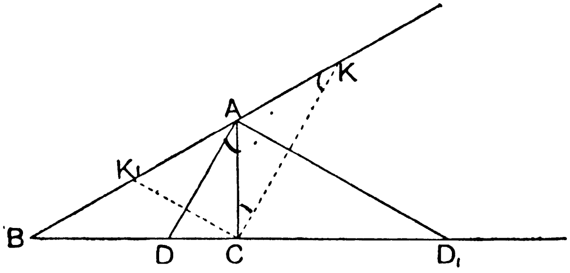
222. ABCD is a parallelogram, and P is taken in BC so that PC is $\frac{1}{3}$ of BC; AC and AP cut the diagonal BD in O and E. Prove that OE is $\frac{1}{10}$ of BD.

223. In the exercise above, if OE is $\frac{1}{8}$ of BD how does P divide BC?

224. ABC is a triangle, and D is the midpoint of BC; any straight line through C meets AD in E and AB in F. Prove that rect. AE.FB = 2 rect. AF.ED.

Proposition XXII. Theorem.

The straight lines bisecting internally and externally the vertical angle of a triangle, divide the base internally and externally into segments which have the same ratio as the adjacent sides of the triangle.



Let the str. lines AD, AD_1 bisect the vertical angle BAC and the external angle CAK at A in any △ ABC, and meet BC in D, D_1:

to prove that $\frac{BD}{DC}=\frac{BA}{AC}=\frac{BD_1}{CD_1}$.

Through C draw CK, CK_1 parl. to AD, AD_1 respectively, meeting AB in K, K_1.

Then ∠ ACK = ∠ CAD [12]

= ∠ DAB = ∠ CKA; [12]

and ∠ ACK_1 = ∠ CAD_1 = ∠ D_1AK = ∠ CK_1A.

∴ $AK=AC=AK_1$. [8]

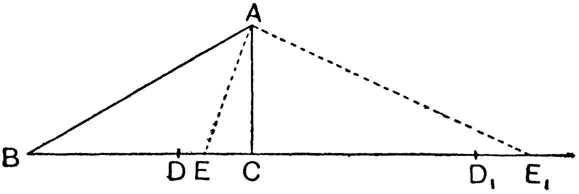
Now DA is parl. to CK,

∴ $\frac{BD}{DC}=\frac{BA}{AK}$ [XXI] $=\frac{BA}{AC}$;

and so also $\frac{BD_1}{CD_1}=\frac{BA}{K_1A}=\frac{BA}{AC}$.

Proposition XXIII. Theorem.

The straight lines drawn from the vertical angle of a triangle to divide the base internally and externally into segments having the same ratio as the adjacent sides of the triangle, bisect the vertical angle internally and externally.



Let the str. lines AD, AD_1, drawn from the vertex A of any △ ABC, divide the base in the ratio of the adjacent sides, so that $\frac{BD}{DC}=\frac{BA}{AC}=\frac{BD_1}{CD_1}$; to prove that AD, AD_1 bisect the angle at A internally and externally.

If AD does not bisect the ∠ BAC, draw AE the bisector of the angle. Then $\frac{BE}{EC}=\frac{BA}{AC}$; [XXII]

but $\frac{BD}{DC}=\frac{BA}{AC}$ [Hyp.]: ∴ $\frac{BD}{DC}=\frac{BE}{EC}$,

i. e. BC is divided internally at D and E in the same ratio. This is impossible [XX];

∴ AD bisects the ∠ BAC. ✓

Similarly can AD_1 be shown to bisect the ∠ BAC externally.

Cor. As $BD:DC=BD_1:CD_1$, it follows that the base of a triangle is divided harmonically at the points where it is met by the internal and external bisectors of the vertical angle.

Exercises.

225. Apply Prop. XXII to divide a given line internally and externally in the ratios 1 :2 and 3 :2.

226. Draw a triangle with sides 20, 15, 18 mm. Find points in the side 18 dividing it in the ratio 4 : 3, and in the side 15 dividing it in the ratio 10 : 9, by means of the Proposition.

227. In the exercise above, find the distance between the two points (of internal and external section) obtained in each side.

*228. Give the construction of Prop. XXII when AB is less than AC; and examine the case of external section of the base when AB = AC.

229. By the help of Props. XXII and XXIII prove that (i) the three internal bisectors, (ii) one internal bisector and two external bisectors, of the angles of a triangle are concurrent.

230. In the figure of Prop. XXIII, $BC=a$, $CA=b$, $AB=c$: find BD, DC, D_1C, and prove that $DD_1=\frac{2abc}{c^2-b^2}$.

*231. In the figure of the same proposition let M be the midpoint of BC: prove that

$$MD : BC = AB - AC : 2\,(AB + AC),$$

and $$MD_1 : BC = AB + AC : 2\,(AB - AC).$$

232. The straight lines bisecting the angles B and C of a triangle ABC meet the opposite sides in D and E: prove that if DE is parallel to BC the triangle is isosceles.

233. From the vertex A of a triangle ABC perpendiculars AK, AL are drawn to the external bisectors of the angles at B, C; prove that KL is parallel to BC and equals the semi-perimeter of the triangle.

234. In a quadrilateral ABCD, the bisectors of the angles at A and C meet on BD: prove that the bisectors of the angles at B and D meet on AC.

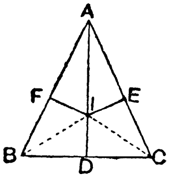
235. The lines trisecting an angle of any triangle cannot divide the opposite side into segments which are all equal.

236. AD is a median of a triangle ABC; the straight lines bisecting the angles ADB, ADC meet the sides AB, AC in P, Q respectively. Shew that PQ is parallel to BC.

237. AD is the bisector of the angle A of a triangle ABC; AC meets the circle described about the triangle ADB in a second point P, and AB meets the circle about the triangle ADC in Q. Prove that BQ is equal to CP.

238. If I is the incentre and J the excentre with regard to the side BC of a triangle ABC, and X the point where IJ meets BC, prove that $\frac{AI}{IX}=\frac{AJ}{JX}=\frac{AB+AC}{BC}$.

239. Prove that the radius of the incircle of an isosceles triangle bears to its altitude the ratio which the base of the triangle has to the sum of its sides.



Solution. Let ABC be the $\triangle$, $AB=AC$. Bisect BC in D, join AD: then the $\triangle$s ABD, ACD are congruent, and AD bisects the $\angle$ A.

$\therefore$ the incentre I will be on AD, and such that BI, CI bisect the $\angle$s B, C respectively.

Hence we have $\frac{ID}{IA}=\frac{DC}{CA}$ [XXII] $=\frac{DB}{BA}=\frac{DC+DB}{CA+BA}$; [vii, § 12]

$$\therefore\ \frac{ID}{IA}=\frac{BC}{CA+BA};\quad \therefore\ \frac{ID}{ID+IA}=\frac{BC}{CA+BA+BC}.$$ [v, *cor.* 1, § 12]

Thus we get $\frac{ID}{AD}=\frac{\text{base}}{\text{perimeter}}$.

240. CD is a chord of a circle and AB the perpendicular diameter; A and B are joined to any point Q in CD, and AQ, BQ produced cut the circle in X, Y respectively. Shew that rect. CX.DY = rect. DX.CY.

§ 14. ELEMENTARY RATIO CONSTRUCTIONS.

Proposition XXIV. Problem.

To divide a given straight line internally and externally in a given ratio.

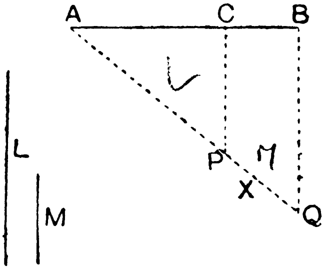


Fig. 1

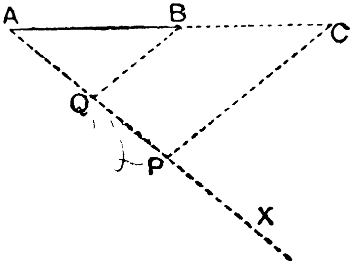


Fig. 2

Let AB be the given str. line, and the ratio of the lengths L, M, the given ratio.

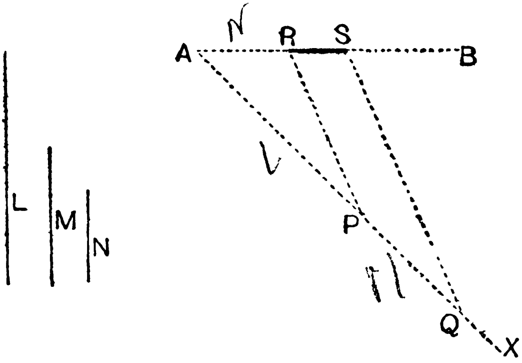
Draw a str line AX to make any convenient angle with AB. Along AX mark off AP = L, PQ = M, the latter being in the *same sense* (or direction) as AP Fig. 1, and in the *opposite sense* to AP in Fig. 2. Join QB and draw PC parl to QB to cut AB or AB produced in C. Then AB shall be divided at C in the given ratio.

$\because$ PC is parl. to QB, $\therefore \frac{AC}{CB} = \frac{AP}{PQ}$ [XXI] $= \frac{L}{M}$.

Cor. By an exactly similar construction we can divide a given straight line similarly to a given divided line, or into any number of parts having given ratios to each other.

Proposition XXV. Problem.

To find the fourth proportional to three given straight lines.



Let L, M, N be the given str. lines. Draw two str. lines AB, AX at any convenient angle. Along AX mark off AP = L and PQ = M, and along AB, AR = N.

Join PR and draw QS parl. to PR to meet AB.

Then RS shall be the fourth proportional.

$\because$ PR is parl. to QS, $\therefore \frac{AP}{PQ} = \frac{AR}{RS}$,

$\therefore \frac{L}{M} = \frac{N}{RS}$, and RS is the fourth proportional.

Cor. By taking N = M, we shall have L : M = M : RS; thus the third proportional to L and M is obtained.

If the lengths L and M are large, PQ = M may be set off in the opposite sense to AP, in order to give the figure conveniently small dimensions.

Exercises.

241. Divide a straight line of length two inches into three parts which shall be to each other as 1 : 4 : 7.

242. Draw a triangle of perimeter 9 cm. with sides as $2 : 4\frac{1}{2} : 5\frac{1}{2}$. Can the sides be proportional to $2 : 3\frac{1}{2} : 6\frac{1}{2}$?

243. Divide a straight line of length $1\frac{3}{4}''$ externally in the ratios 2 : 9 and 9 : 2. Measure the parts and verify by calculation.

244. A straight line APB is such that $AP = 1\cdot75'$, $PB = \cdot25'$. Divide it externally at Q so that $AQ:BQ = AP:PB$, and by measurement show that $\frac{1}{AP}+\frac{1}{AQ}=1$.

*245. A straight line AB is divided internally and externally at P and Q in the same ratio, and M is the midpoint of AB : prove that rect. $MP.MQ = MA^2$.

246. Construct the second figure of Prop. XXIV when M is greater than L, and examine the case when $M = L$.

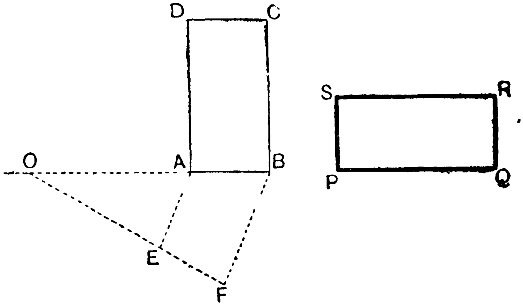
247. Find x geometrically in the following proportions, and verify the answers by calculation :—

(1) $1\cdot2:3=5:x$; (2) $1\cdot2:3=x:5$; (3) $x:2\cdot5=2\cdot5:6$.

248. Find by geometrical construction the values of $\frac{18}{3\cdot9}$ and $\frac{1\cdot7\times5\cdot3}{4\cdot5}$.

249. The sides BC, CA, AB of a triangle are of lengths a, b, c respectively: find by construction the fourth proportionals to (1) a, b, c; (2) c, b, a; (3) a, c, b.

250. On a given base draw a rectangle equal in area to a given rectangle.



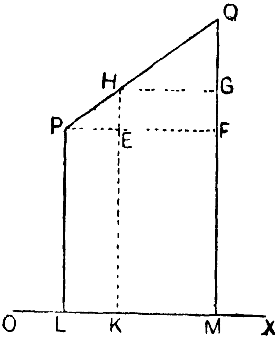
Solution. Let PQ be the given base and ABCD the given rectangle. Produce BA, take $AO = PQ$; draw any line OEF, take $OE = AD$. Join AE, draw BF parl. to AE· then EF is the other side of the rectangle.

Draw QR perp. to PQ and $= EF$, and complete PQRS, which is the required figure. As we require $PQ.QR = AB.AD$; $\therefore \frac{PQ}{AB}=\frac{AD}{QR}$, so that QR (or EF) is the fourth proportional to PQ, AB, AD: hence the construction.

251. On a base $1\frac{1}{5}''$ long construct a rectangle equal in area to a square of side $1\frac{4}{5}''$. Measure the other side of the rectangle and verify the result.

252. The sides of a triangle are 17, 21, 24. Construct on a base of length 30 an equivalent rectangle, drawing the figure to any suitable scale.

*253. Two points P,Q, on the same side of a line OX, are at distances y_1, y_2, from it: if H is taken in PQ so that PH : HQ $= c : d$, prove that the distance of H from OX is $\frac{y_1 d + y_2 c}{c+d}$.



Solution. Draw the distances PL, QM, HK from the given line OX, and PEF, HG parl. to it; let HK $= y$.

We have $\frac{\text{FG}}{\text{GQ}} = \frac{\text{PH}}{\text{HQ}} = \frac{c}{d}$;

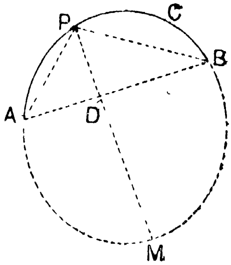
$\therefore$ $\frac{\text{HE}}{\text{GQ}} = \frac{c}{d}$, *i. e.*, $\frac{y - y_1}{y_2 - y} = \frac{c}{d}$.

Multiply out and solve for y.

254. Solve the problem above when (1) P and Q are on opposite sides of OX, (2) H is in PQ produced.

*255. The centroid, G, of three points P, Q, R, is thus constructed: bisect PQ in A, and divide AR in G so that AG:GR $= 1:2$. If x_1, x_2, x_3 are the distances of P, Q, R from a given straight line, find the distance of G from the same line.

256. ACB is a given arc of a circle: find a point P on it such that the chords AP, PB may be in a given ratio.



Solution. Complete the ○ of which ACB is the arc; bisect the remaining arc at M [57], and divide the chord AB in the given ratio, say 2 : 3 [XXIV]. Join M to the pt. of division, D, and let MD meet ACB in P.

$\because$ arc AM = arc BM, $\therefore$ $\angle$ APD $= \angle$ BPD [38, 44];

$\therefore$ AP : PB = AD : DB [XXII] = 2 : 3 [*Cons.*].

257. Through a given point O within a given angle BAC draw a straight line XOY terminated by AB and AC, and such that XO . OY is equal to a given ratio.

CHAPTER VI.

SIMILAR FIGURES · CENTRES OF SIMILARITY.

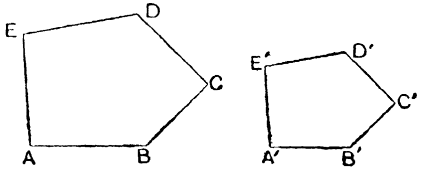
§ 15. SIMILAR TRIANGLES.

Def. Two rectilineal figures are said to be *equiangular to one another* when the several angles of one taken in order are equal respectively to the several angles of the other taken in the same order.

More briefly such figures are called *equiangular*; and this term does not necessarily imply equality among the angles of either figure.

The angles which are equal in pairs are said to be *corresponding* angles of the figures, and a side of one is said to be *corresponding* or *homologous* to a side of the other, when the two sides terminate in pairs of corresponding angles.

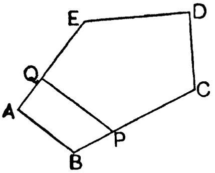
Two rectilineal figures are said to be similar when they are *equiangular to one another* and have also their *corresponding sides proportional.*



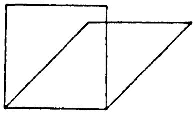
Thus in the accompanying figure the two polygons are similar (i) if $\angle A = \angle A'$, $\angle B = \angle B'$, ..., $\angle E = \angle E'$, and (2) if also $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \dots = \frac{EA}{E'A'}$.

Alternando we have $\frac{EA}{AB} = \frac{E'A'}{A'B'}$, $\frac{AB}{BC} = \frac{A'B'}{B'C'}$, ...; hence the sides about each of the equal angles are successively in proportion.

Both conditions of similarity are essential; but it will be seen from the immediately following propositions that either condition alone is sufficient for triangles. This is due to the fact that *the triangle is the simplest of rectilineal figures.*



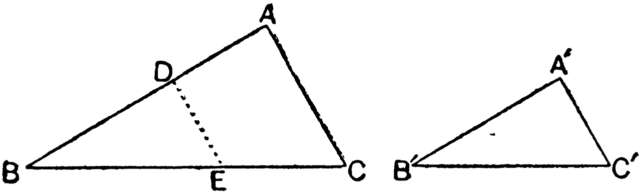
In the case of other figures both are necessary. For instance, the annexed diagram shows two pentagons which are equiangular but not similar, as $\frac{BC}{CD}$, $\frac{PC}{CD}$ are not equal ratios.



Again a square and a rhombus have their sides in proportion, but are not equiangular; they are therefore not similar.

Proposition XXVI. Theorem.

If two triangles are equiangular to one another, their corresponding sides are proportional, and the triangles are similar.



Let the $\triangle$s ABC, A'B'C' be equiangular, so that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$: to prove that they are similar.

Apply the $\triangle$ A'B'C' to ABC so that $\angle B'$ coincides with the equal $\angle B$ and A' falls along BA (produced if necessary) at some pt. D; then C' falls along BC at E, so that DE is the new position of A'C'.

$\because \angle D = \angle A' = \angle A$, $\therefore$ DE is parl. to AC.

Hence $\frac{AB}{BD} = \frac{CB}{BE}$ [XXI, *Cor. 1*], *i.e.*, $\frac{AB}{A'B'} = \frac{CB}{C'B'}$.

Similarly, by applying the $\triangle$ A'B'C' to ABC so as to make $\angle C'$ coincide with $\angle C$, it may be proved that $\frac{BC}{B'C'} = \frac{AC}{A'C'}$.

$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$;

also the $\triangle$s are equiangular, [*Hyp.*]

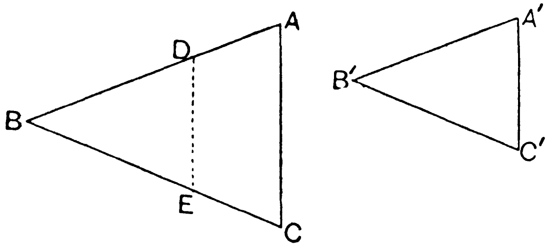
$\therefore$ they are similar. [*Def.*]

Cor. 1. Two triangles are similar if two angles of one are severally equal to two of the other; for then the third angles are equal.

Cor. 2. Two right-angled triangles are similar if an acute angle of one is equal to an acute angle of the other.

Proposition XXVII. Theorem.

If two triangles have their sides proportional when taken in order, their angles opposite to corresponding sides are equal, and the triangles are similar.



Let the Δs ABC, $A'B'C'$ have their sides proportional in order, so that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$: to prove that they are similar.

From BA, BC (produced if necessary) cut off $BD = B'A'$, $BE = B'C'$, and join DE.

$\because \frac{BA}{B'A'} = \frac{BC}{B'C'}$, $\therefore \frac{BA}{BD} = \frac{BC}{BE}$,

$\therefore$ DE is parl. to AC. [XXI, ii]

Hence the Δs BAC, BDC are equiangular, and $\therefore$ similar. [XXVI]

$\therefore \frac{BD}{DE} = \frac{BA}{AC} = \frac{B'A'}{A'C'}$ [*Hyp.*];

but $BD = B'A'$ [*Cons.*], $\therefore$ $DE = A'C'$.

$\therefore$ the Δs BDE, $B'A'C'$ are congruent [5];

$\therefore$ their corresponding angles are equal.

But the Δ BAC is equiangular to BDE [*Proved*];

$\therefore$ the Δs BAC, $B'A'C'$ are equiangular :

also their sides are proportional in order [*Hyp.*],

$\therefore$ they are similar. [*Def.*]

Exercises.

258. Construct a triangle of sides $51\frac{1}{2}$, $60\frac{1}{2}$, 72 mm., and measure its angles. Hence find the other sides of a triangle whose largest side is 18 and whose angles are 45°, 55°, 80°.

259. Construct two triangles of different size whose sides are as $1 : 3 : 3\frac{2}{3}$; show by measurement that they are equiangular.

260. Any point D in the base BC of a triangle is joined to the vertex A, and AD is divided at O so that $AO : OD = 5 : 7$; POQ is drawn parallel to the base and terminated by the sides of the triangle. If $BC = 1{\cdot}2''$, $CA = {\cdot}8''$, $AB = 1{\cdot}4''$, find PQ by measurement and verify by calculation.

261. Draw two right-angled triangles each having an acute angle of 48^0 ; measure the ratio of the hypotenuse to the side opposite the given angle in each triangle.

262. Draw two straight lines inclined at an angle of 70^0, and find a series of points within the angle whose distances from the lines are in the ratio 3 : 4.

263. Within a triangle ABC find a point P such that its distances from BC, CA, AB, are as 7 11·16 respectively, and measure BP and CP.

264. A man 5 ft. 8 in. high observes that his shadow cast by the sun is 1 yd. 4 in. : calculate the height of a tower which casts a shadow of 357 ft. at the same instant.

265. Draw two circles of radii $1''$ and 2 cm.; circumscribe triangles of angles 50^0, 55^0, 75^0 about each, and compare their corresponding sides.

266. Prove the results of (VIII, *Cor.* 1) and (IX) by means of similar triangles.

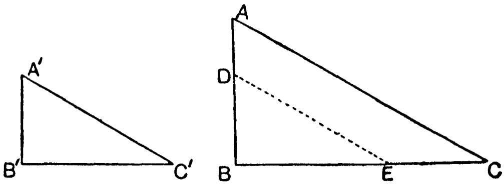
267. By help of Prop. XXVI prove that the medians of a triangle concur at a point of trisection of each.

268. In a triangle ABC, AD and CE are the perpendiculars to the bisector of the angle ABC : shew that $AD.BE = CE.BD$.

*269. In two similar triangles prove that two corresponding sides are proportional to (1) the altitudes drawn to them, (2) the bisectors of the opposite angles terminated by them, (3) the radii of the circumcircles.

*270. Two right-angled triangles are similar if the hypotenuse and one side of one are proportional to the hypotenuse and one side of the other.

Solution. In the Δs ABC, A′B′C′, rt.-angled at B, B′, let $AC : A'C' = AB : A'B'$. Suppose $BA > B'A'$; from BA cut off $BD = B'A'$, and draw DE parl. to AC.



∵ the Δs ABC, DBE are equiangular,

$\therefore \frac{AC}{DE} = \frac{AB}{BD} = \frac{AB}{B'A'}$ [*Cons.*] $= \frac{AC}{A'C'}$ [*Hyp.*];

$\therefore$ DE = A′C′. Also DB = A′B′, $\angle$B = $\angle$B′,

$\therefore$ the Δs DBE, A′B′C′ are congruent [7, *a*], and equiangular.

But the Δs DBE, ABC are equiangular ;

$\therefore$ the Δs ABC, A′B′C′ are equiangular, and hence similar.

271. Two circles touch at a point A, and any straight line through A cuts them at P and Q: prove that AP, AQ are in the ratio of the corresponding diameters.

272. Two circles intersect at A and B, and any straight line through A cuts them again at P and Q: prove that BP, BQ are in the ratio of the corresponding radii.

273. A chord BC of a circle cuts off an arc whose midpoint is A ; any straight line APQ cuts BC in P and the circle again in Q. Prove that rect. AP.AQ = sq. on AC.

274. The straight line joining the midpoints of the parallel sides of any trapezium passes through the point of intersection of the diagonals ; and the straight line through this point parallel to the parallel sides and terminated by the other sides, is bisected at the point.

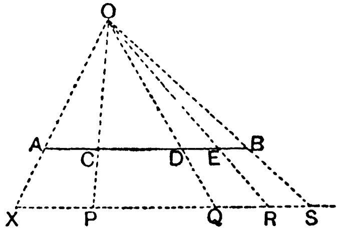
275. In the parallel sides AD, BC, of a trapezium the points P, Q, are taken such that AP . PD = BQ : QC : prove that AB, CD, PQ are concurrent.

276. ABCD is a quadrilateral having right angles at A and C, and BK, DL are drawn perpendicular to AC : prove that AK = CL.

*277. Four straight lines AK, BL, CM, DN meet in a point O, and ABCD, KLMN are parallel straight lines cutting them : prove that AB : BC : CD = KL : LM : MN.

278. Apply the exercise above to divide a given straight line into n parts having given ratios to each other.

Solution. Let AB be the given str. line, and suppose it is required to divide it into four parts proportional to a, b, c, d.



Take a pt. X and draw XS parl. to AB ; with any convenient unit mark off XP, PQ, QR, RS of lengths a, b, c, d units respectively. Let XA, SB meet in O ; join OP, OQ, OR to cut AB in C, D, E respectively.

It may be proved, as in the previous exercise, that AC : CD : DE : EB = XP : PQ : QR : RS = $a : b : c : d$.

279. AB, AC are two chords of a circle, and AT is a tangent; CD is drawn parallel to AT to meet AB in D. Prove that AD is the third proportional to AB and AC.

280. A square is inscribed in a right-angled triangle with one side on the hypotenuse and the two opposite angular points on the two sides of the triangle: prove that the three segments of the hypotenuse are in continued proportion.

*281. X, Y are points on the sides AB, AC of a triangle ABC, such that XY is parallel to BC: find the locus of the point of intersection of BY and CX.

*282. A point P moves so that its distances from two fixed straight lines AB, AC are in a given ratio: prove that the locus of P is a straight line passing through A.

283. ABC, DBC are two triangles of equal area on the same side of the base BC: prove that the parts, intercepted within the triangles, of any straight line parallel to BC are equal.

*284. From a point O on the circumference of a circle perpendiculars OP, OQ, OR are drawn on two tangents AC, BC, and their chord of contact respectively: prove that rect. $OP.OQ = OR^2$.

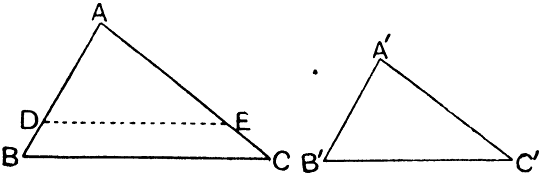
*285. O is a fixed point and XY a fixed straight line; lines are drawn from O to various points of XY and divided in a fixed ratio. Find the locus of the points of section.

*286. O is a fixed point in the plane of a fixed circle of radius r; lines are drawn from O to various points of the circumference and divided in the fixed ratio $l : m$. Prove that the locus of the points of section is a circle of radius $\frac{l}{l+m}r$.

287. Through one of the points of intersection of two circles draw a chord which shall be divided at the point in the ratio of the radii.

Proposition XXVIII. Theorem.

If two triangles have one angle of one equal to one angle of the other and the sides about these angles proportional, the triangles are similar.



Let the $\triangle$s ABC, A′B′C′ have $\angle A = \angle A'$, and AB : A′B′ = AC : A′C′: to prove that they are similar.

Apply the $\triangle A'B'C'$ to ABC so that the equal angles A', A coincide, and B' falls on AB at D; then C' falls on AC at E, and DE is the new position of $B'C'$.

$\because \frac{AB}{A'B'} = \frac{AC}{A'C'}$, $\therefore \frac{AB}{AD} = \frac{AC}{AE}$,

$\therefore$ DE and BC are parl. [XXI, ii]

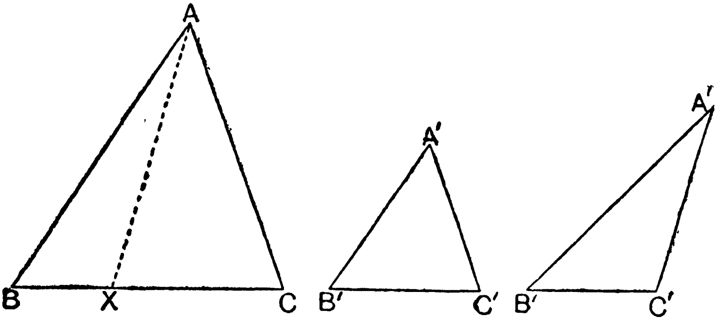
$\therefore \angle B' = \angle D = \angle B$, $\angle C' = \angle E = \angle C$. [12]

Hence the $\triangle$s ABC, $A'B'C'$ are equiangular ;

$\therefore$ they are similar. [XXVI]

Proposition XXIX. Theorem.

If two triangles have two sides of one proportional to two sides of the other and have likewise the angles opposite to one corresponding pair of these sides equal, then the angles opposite to the other corresponding pair are either equal or supplementary; and in the former case the triangles are similar.



Let the $\triangle$s ABC, $A'B'C'$ have $\angle B = \angle B'$ and also $AB : A'B' = AC : A'C'$: to prove that $\angle C = \angle C'$ or $\angle C + \angle C' = 2$ rt. angles.

(i) If $\angle A = \angle A'$, then $\angle C = \angle C'$; the $\triangle$s are equiangular, and therefore similar.

(ii) But if $\angle A$ is not $= \angle A'$, suppose $\angle A$ greater ;
cut off $\angle BAX = \angle B'A'C'$.

The $\triangle$s BAX, $B'A'C'$ are equiangular, and therefore similar.

Hence $\frac{AB}{A'B'} = \frac{AX}{A'C'}$ [XXVI] ; but $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ [*Hyp.*] ;

$\therefore \frac{AX}{A'C'} = \frac{AC}{A'C'}$, *i.e.*, $AX = AC$.

$\therefore \angle ACX = \angle AXC$ [8],

$= \text{suplt. of } \angle AXB = \text{suplt. of } \angle A'C'B'$.

$\therefore \angle C + \angle C' = 2$ rt. angles.

Cor. If the given equal angles B, B′ are right, the angles C, C′ are necessarily acute [10], and cannot be supplementary; they are therefore equal. Hence if the given angles are both right, the triangles are similar. [See Exer. 270.] So also, if the given angles are both obtuse the triangles are similar.

The student should notice the close correspondence of the conditions which determine the congruence with those which determine the similarity of triangles. The chief difference is that in the case of similarity no linear equality is given; for neither lengths nor areas are necessarily equal, the triangles being only required to be of the same shape. We thus have—

(*a*) two sides *equal* and included angles equal [4], two sides *proportional* and included angles equal [XXVIII];

(*b*) three sides *equal* [5], three sides *proportional* [XXVII];

(*c*) two angles equal and one side *equal* [6], two angles equal and therefore the third angles also equal [XXVI]; and

(*d*) two sides *equal* and angles opposite to one pair of them equal [7], two sides *proportional* and angles opposite to one pair of them equal [XXIX].

Exercises.

288. Construct three triangles each having two sides in the ratio 3 : 4, and the angle opposite to the smaller of these an angle of 32^0. Are all such triangles similar?

289. Construct as many different kinds of triangles as possible each having two sides in the ratio 5 : 3 and an angle of 32^0 opposite to the larger of these.

290. Two straight lines AB, CD intersect at O so as to make rect. AO.OB = rect. CO.OD: prove by Prop. XXVIII that the points A, C, B, D are concyclic.

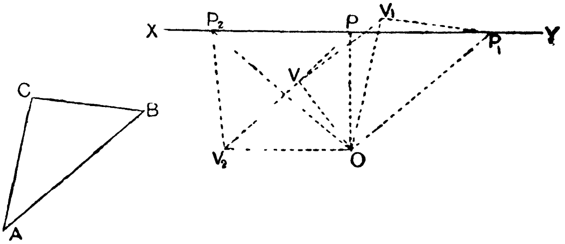
291. ABC, A′B′C′ are similar triangles, AD and A′D′ being corresponding medians: prove that AD : A′D′ = BC : B′C′.

292. In the exercise above if D, D′ are points in BC, B′C′ such that BD : DC = B′D′ : D′C′, prove that the result still holds good.

293. OABC...is a straight line; through A, B, C,...parallel straight lines AP, BQ, CR..., in the same direction, are drawn so as to make AP : BQ : CR... = OA : OB : OC.... Prove that the points O, P, Q, R...are collinear.

294. AB is the diameter and C the centre of a semicircle; N and T are points in CB and CB produced such that $AT : AC = AN : CN$, and TP is the tangent from T. Prove that the $\angle$ CNP is right.

*295. O is a fixed point and XY a fixed straight line; on the lines drawn from O to various points of XY triangles similar to a given triangle are similarly described. Find the locus of the vertices of the triangles.



Solution. Let ABC be the given Δ. Draw OP perp. to XY, and describe on it the Δ OPV equiangular to ABC. Then the Δ OPV is similar to ABC, also OP is a fixed line and V a fixed pt.

Take other pts. P_1, P_2,...on XY, and draw similarly the Δs OP_1V_1, OP_2V_2,...equiangular to ABC, and therefore similar to it.

Join VV_1, VV_2,...

The Δs OPV, OP_1V_1 are equiangular, and therefore similar; hence $\frac{OP}{OP_1} = \frac{OV}{OV_1}$. Also $\because$ $\angle VOP = \angle V_1OP_1$, $\therefore$ $\angle VOV_1 = \angle POP_1$. Hence in the Δs OVV_1, OPP_1, two sides of the first are proportional to two of the second, and the included angles are equal;

$\therefore$ these Δs are similar [XXVIII].

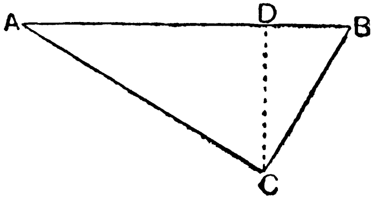
$\therefore$ $\angle OVV_1 = \angle OPP_1 =$ one rt. angle.

Thus V_1 lies on a str. line through V perp. to OV, *i.e*, on a fixed str. line. Similarly V_2 and the other vertices are seen to lie on the same fixed line, which is therefore the required locus.

§ 16. MEAN PROPORTIONAL.

Proposition XXX. Theorem.

In a right-angled triangle the perpendicular drawn from the right angle to the hypotenuse divides the triangle into two triangles similar to the given triangle and to one another.



Let ABC be a △ rt.-angled at C, draw CD perp. to AB : to prove that the △s ADC, BDC are similar to the △ ABC and to one another.

In the △s ADC, ABC, ∠ A is common; ∠ ADC = ∠ ACB, each being right; and ∠ ACD = ∠ ABC, each being the complt. of ∠ A.

Hence the △s are equiangular, and therefore similar.

In the same way the △s BDC, ABC may be proved similar.

∴ the △s ADC, BDC, being similar each to the △ ABC, are similar to one another.

Cor. 1. ∵ △s ADC, BDC are similar, ∴ AD : DC = CD : DB [XXVI]; ∴ rect. AD.DB = CD^2 [§ 12, i].

Hence, *if a perp. is drawn from the right angle of a rt-angled △ to the hypotenuse, the square on the perpendicular is equal to the rectangle of the segments of the hypotenuse.* [See V, *Cor.*, and Exer. 59.]

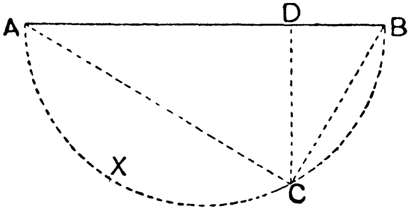
Cor. 2. ∵ △s ADC, ABC are similar, ∴ AC : AB = AD : AC ;
∴ rect. AD.AB = AC^2. So also rect. BD.BA = BC^2.

Hence, *the square on either side of the rt-angled triangle is equal to the rectangle of the hypotenuse and the adjacent segment of it made by the perpendicular from the right angle.* [See Exer. 60.]

The student should see that the three triangles in the figure are really similar, though they are not similarly situated. Thus the △ BDC may be made to occupy a position like that of the △ ADC by being turned in its plane through a right angle; the △ ABC will further require to be turned about AB as axis.

Proposition XXXI. Problem.

To find the mean proportional to two given straight lines.



Place the given lines, AD, DB, end to end in a straight line as ADB; on AB describe the semi-◯AXB, and draw DC perp. to AB to meet it in C · then DC is the mean proportional. Join CA, CB.

∵ ∠ ACB = rt. angle [46], and CD is perp. to AB,

∴ CD is the mean proportional to AD, DB [XXX, *Cor.* 1]

When the given str. lines are so large that the length ADB is inconvenient to draw, they should be placed as AB, AD, so that AD falls on AB; making the same construction, AC will now be found to be the mean proportional.

The proof gives another solution of Prop. V; the student should read the note on that problem, and solve Exers. 43, 46 and 47.

Exercises.

296. In the figure of Prop. XXXI, AC = 2·4″, BC = ·7″; calculate AD, BD, CD.

297. One side of a rectangle is 13 mm. long; find the other side, if the area is the same as that of an equilateral triangle of side 2 cm.

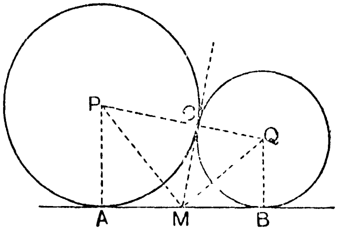
298. Draw a rectangle equal in area as well as perimeter to a triangle of sides 1·5, 2, 2·5 cm., and measure its sides.

299. Divide a given straight line (i) externally, (ii) internally, so that the rectangle contained by the segments is equal to the square on another given straight line.

300. In a triangle ABC, right-angled at C, the prependicular CD drawn to AB divides it in extreme and mean ratio so that AB : AD = AD : DB : prove that AC is the mean proportional to AB and BC.

*301. Two parallel tangents of a circle meet a third tangent in A and B : if T is the point of contact of this tangent, prove that rect. AT.TB = sq. of the radius.

302. Two circles touch externally and an exterior common tangent meets them at A and B : prove that AB is the mean proportional to the diameters of the circles.



Solution. Let P, Q be the centres of the ○s, C their pt. of contact; draw CM the common tangent at C to meet AB; join MP, MQ, AP, BQ.

As MC, MA are tangents to the same ○,

∴ $MC = MA$ and $\angle AMP = \angle CMP$.

So also $MC = MB$ and $\angle BMQ = \angle CMQ$ [42].

Hence $CM = \frac{1}{2} AB$, and $\angle PMQ = \angle PMC + \angle CMQ$

$= \frac{1}{2}(\angle AMC + \angle BMC) =$ one rt. angle.

Now MC is the perp. of the rt.-angled △MPQ ;

∴ MC is the mean proportional to CP, CQ, the radii.

Hence AB, *i. e.*, 2 MC, is the mean proportional to the diameters of the ○s.

*303. From any point T outside a circle tangents TP, TQ are drawn to the circle, and O is its centre : if PQ meets OT in N, prove that ON, OP, OT are in continued proportion.

304. AB is the diameter of a semicircle and BP the tangent to it at B; any straight line through A meets the semicircle in Q and BP in P. Prove that AQ, AB, AP are in continued proportion.

305. AB is the diameter of a semicircle and P any point on its arc; a straight line perpendicular to AB at C meets AP, BP (produced if necessary) in D, E, and the arc in F. Prove that CD : CF = CF : CE.

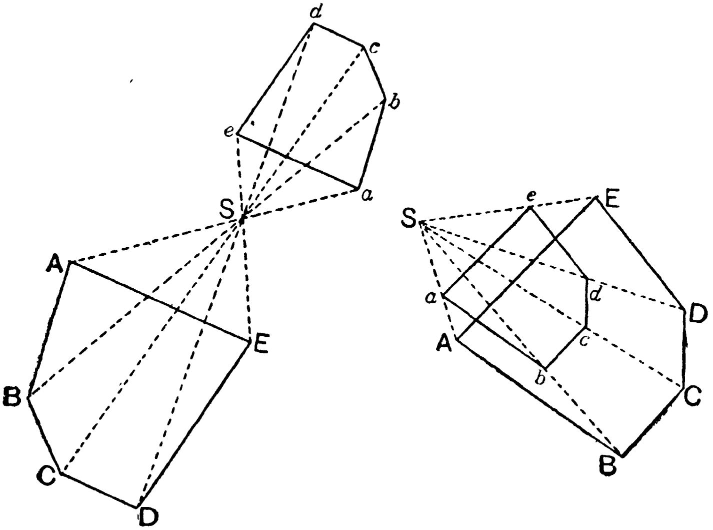
306. In the exercise above if the line CDE falls outside the semicircle, prove that rect. CD.CE = sq. of the tangent from C to the circle.

307. Construct an isosceles triangle on a given base so as to have a given area.

§ 17. CENTRES OF SIMILARITY.

Proposition XXXII. Theorem.

If two unequal similar polygons are placed so that their corresponding sides are parallel, the lines joining their corresponding vertices all meet in a point, whose distances from any two corresponding vertices are in the ratio of the corresponding sides of the polygons.



Let ABCDE, *abcde* be similar unequal polygons. If *ab* is placed so as to be parl. to AB, then *bc* is parl. to BC, as $\angle B = \angle b$; so also the other corresponding sides are parl.

To prove that A*a*, B*b*, ..E*e* are concurrent.

Let A*a*, B*b* meet in S; join SC, S*c*.

The $\triangle$s ABS, *ab*S are equiangular;

$\therefore \frac{AB}{ab} = \frac{BS}{bS}$ [XXVI]. But $\frac{AB}{ab} = \frac{BC}{bc}$ [*Def.*],

$\therefore \frac{BS}{bS} = \frac{BC}{bc}$; and $\angle SBC = \angle Sbc$ [12].

$\therefore$ the $\triangle$s SBC, S*bc* are similar. [XXVIII]

Hence $\angle BSC = bSc$; $\therefore$ SC, S*c* are in one str. line, *i.e.*, the line C*c* goes through S.

So also S, D, *d* and S, E, *e* are collinear:

$\therefore$ the lines joining corresponding vertices concur in the pt. S.

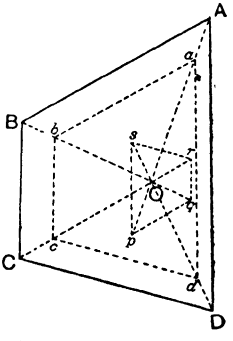
Again, from the similar $\triangle$s SAB, Sab, $\frac{SA}{Sa}=\frac{AB}{ab}$; so also $\frac{SB}{Sb}=\frac{SC}{Sc}=\frac{BC}{bc}=\frac{AB}{ab}$ [*Def.*], and so on.

$\therefore$ the distances of S from corresponding vertices are in the ratio of any two corresponding sides of the polygons.

Cor. If the straight lines joining any point to the vertices of a polygon are divided, all internally or all externally, in the same ratio and order, the points of division form the vertices of a polygon similar to the given one and the corresponding sides of the two are parallel.

Defs. The polygons ABCDE, *abcde* of the Proposition are said to be *similarly situated* or *homothetic*, the point S, where the lines joining corresponding vertices concur, is said to be a *centre of similarity* or *homothetic centre* of the two figures. In the first figure, S, which divides Aa, Bb,...internally, is called the *internal* or *inverse* centre of similarity; when S lies in Aa, Bb,...produced, as in the second figure, it is called the *external* or *direct* centre of similarity.

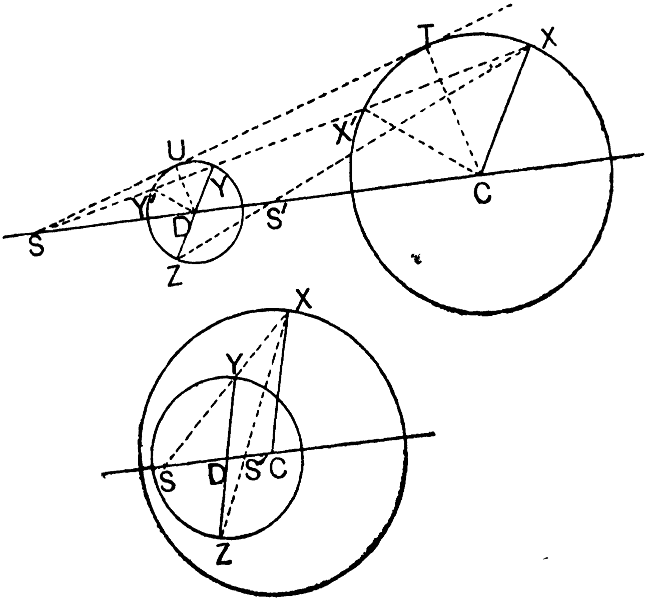
In either case S may be within both figures. Thus take any point O within the figure ABCD; divide OA, OB, OC, OD internally in the same ratio at a, b, c, d: then $abcd$ is similar and *similarly situated* with ABCD. It will be seen that AB and ab, BC and bc, ... are drawn parallel in the same direction. But suppose AO, BO, CO, DO, are produced to p, q, r, s, so that Op: pA = Oq:qB = Or:rC = Os:sD: then $pqrs$ is still similar to ABCD, but is now *oppositely situated*, so that, although AB, pq are parallel, they are drawn from A and p respectively in opposite directions. In the first case, O is the *direct centre* of similarity, in the second the *inverse centre*. The ratio Oa:OA or Op:OA, with its proper sign, is called the *ratio of similarity* of the figures.



If the two polygons are regular figures of the same number of sides, they are necessarily similar: if then they are placed with their corresponding sides parallel, they will have a centre of similarity, direct or inverse. Let circles be described about the figures; and suppose the circles remain fixed while the sides of the figures increase in number and diminish in magnitude, the figures remaining regular all through. When this process is continued indefinitely, the limiting forms of the regular polygons will be the circumcircles themselves. Hence it is seen that two circles have also two centres of similarity: an independent proof of this fact is given in the following proposition.

Proposition XXXIII. Theorem.

The straight lines joining the extremities of any two parallel radii of two circles pass through one or the other of two fixed points on the line of centres of the circles.



Let C, D be the centres of two given ○s, CX any radius of the first; draw YDZ the parl. diameter of the second, and join XY, XZ, cutting the line of centres, CD, in S, S′: to prove that S and S′ are fixed points.

From the similar △s CSX, DSY,

$\frac{CS}{SD}=\frac{CX}{DY}=$ a fixed ratio (that of the radii);

∴ S divides the fixed line CD externally in a fixed ratio, and therefore is a fixed pt. [XX]

So also S′ divides the fixed line CD internally in the ratio of the radii, and is therefore a fixed pt.

Cor. 1. The ⊙ whose centre is C and the pt. S or S′ being given, Y or Z may be obtained by joining SX, S′X, and dividing these internally or externally in a constant ratio: hence, if the straight lines joining any fixed point to points on the circumference of a given circle are divided, all internally or all externally, in the same ratio and order, the points of division lie on another circle, the corresponding radii of the two being parallel. [See Exer. 286.]

Cor. 2. If common tangents to the two circles can be drawn, it is seen that the outer common tangents pass through S and the inner through S′.

Def. The points S and S′ in the figure of the Proposition are called the *centres of similitude* (or *similarity*) of the two circles; S is the *external* or *direct* centre of similitude and S′ the *internal* or *inverse*. It should be noted that SY:SX is a positive ratio, while S′Z:S′X is negative.

In the first figure of the Prop. let the outer common tangent, TU, be drawn; it divides CD externally in the ratio of the radii and, therefore, goes through S. Let SXY cut the ⊙s again in X′, Y′; join CX′, DY′. Then ∠ DY′Y = ∠ DYY′ = ∠ CXX′ = ∠ CX′X. Thus CX′, DY′ are parallel radii, as was to be expected since the line joining their extremities goes through S. We have also $SX.SX' = ST^2$, $SY.SY' = SU^2$ [IX].

As the pairs of △s SCT and SDU, SCX and SDY, SCX′ and SDY′ are similar,

$$\therefore \quad \frac{ST}{SU} = \frac{SX}{SY} = \frac{SX'}{SY'}, \text{ each being } = \frac{CT}{CU}.$$

From $ST^2 = SX.SX'$ and the equal ratios given above,

we get $\dfrac{ST}{SX'} = \dfrac{SX}{ST} = \dfrac{SY}{SU}$;

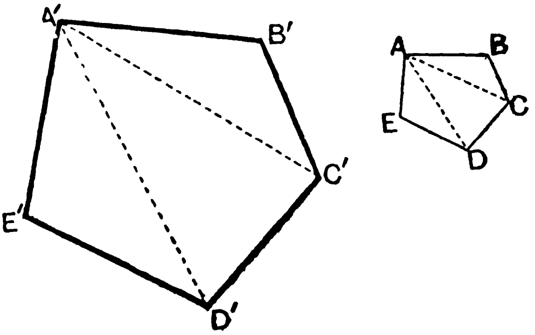
$$\therefore \quad SX'.SY = ST.SU.$$

Similarly we have $SX.SY' = ST.SU$.

∴ the products of the distances of X and Y′ and of X′ and Y from S are constant. The result is seen to be true for any line through S or S′ cutting both circles; and holds good even when the ⊙s have no common tangents as in the second figure. In the latter case, a line through S at right angles to CD cuts the two ⊙s in T and U so that $SX.SX' = ST^2$, $SY.SY' = SU^2$ [VIII, *Cor.* 3].

Proposition XXXIV. Problem.

On a given straight line to construct a polygon similar to a given polygon.



Let A′B′ be the given str. line and ABCDE the given polygon. Join one angular pt. of the polygon, A, to the opposite pts. C, D. At A′ and B′ make the ∠s B′A′C′, A′B′C′ respectively equal to BAC, ABC, thus obtaining C′; at A′ and C′ make the ∠s C′A′D′, A′C′D′ respectively equal to CAD, ACD; and at A′, D′ the ∠s D′A′E′, A′D′E′ equal to DAE, ADE. Then A′B′C′D′E′ shall be the required polygon.

It is readily seen that the several ∠s of A′B′C′D′E′ are equal to the corresponding ∠s of ABCDE.

The △s A′B′C′, ABC are equiangular; ∴ $\frac{A'B'}{AB} = \frac{B'C'}{BC}$.

Also $\frac{B'C'}{BC} = \frac{C'A'}{CA}$; but from the similar △s A′C′D′, ACD, we have $\frac{C'A'}{CA} = \frac{C'D'}{CD}$: ∴ $\frac{B'C'}{BC} = \frac{C'D'}{CD}$.

In the same way we can prove $\frac{C'D'}{CD} = \frac{D'E'}{DE} = \frac{E'A'}{EA}$.

Thus $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$,... $\frac{E'A'}{EA}$ are all equal, and the sides about the equal angles are proportional:

∴ the figure A′B′C′D′E′ is similar to ABCDE.

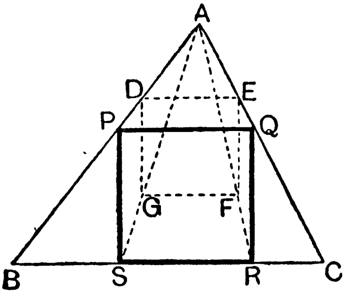
Cor. 1. If A′B′ is placed parallel to AB, the above construction will make A′B′C′D′E′ similar and homothetic to ABCDE. In this case we can also proceed thus: join AA′, BB′, meeting in O; on OC, OD, OE take pts. C′, D′, E′ so that $\frac{OC'}{OC}=\frac{OD'}{OD}=\frac{OE'}{OE}=\frac{OA'}{OA}$ or $\frac{OB'}{OB}$; then A′B′C′D′E′ is the required polygon [XXXII, *Cor.*].

Cor. 2. Similar polygons can be divided into the same number of similar triangles by joining corresponding vertices in the figures.

From propositions XXXII, XXXIV, and their corollaries, it will be seen that a rectilineal figure similar to a given one can always be constructed so that the two have a given point as a centre of similarity. This fact may often be utilized in the inscription or circumscription of figures of given shape in and about a given figure. An illustration is given in the following proposition; others will be found in the exercises.

Proposition XXXV. Problem.

To inscribe a square in a given triangle: that is, to construct a square one side of which lies along one side of the triangle, and the two opposite angular points fall severally on the remaining sides of the triangle.



Let ABC be the △, and BC its side along which a side of the required sq. is to lie. Draw any str. line parl. to BC meeting AB, AC in D, E; on DE draw the sq. DEFG [55]; join AF, AG, meeting BC in R, S; draw RQ, SP perp. to BC to meet AC, AB. Then PQRS is the required square.

∵ QR, EF are perp. to parl. str. lines BC, DE,

∴ they are parl.; so also are PS and DG.

Hence AE : EQ = AF : FR = AG : GS = AD : DP;

∴ PQ is parl. to DE [XXI], *i.e.*, to BC.

And the $\angle$s RSP, QRS are rt. angles, $\therefore$ the $\angle$s SPQ, PQR are also right, and the figure PQRS is rectangular.

Again $\because$ the $\triangle$s AEF and AQR, AFG and ARS, AGD and ASP are similar,

$$\therefore \frac{EF}{QR} = \frac{AF}{AR} = \frac{FG}{RS} = \frac{AG}{AS} = \frac{DG}{SP};$$

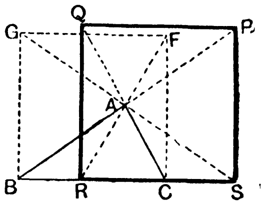
but EF = FG = GD ; [*Cons.*]

$\therefore$ QR = RS = SP ; and, $\because$ PQRS is a parlm., PQ = QR.

Thus the figure is equilateral ;

$\therefore$ it is a square.

Instead of DE we can take BC itself, and on it describe a square outwardly. It is evident that there are two other solutions, in which a side of the square lies on CA, AB successively. Moreover, the points P, Q, instead of falling on AB, AC, may fall on these lines produced ; in this case, the square on DE or BC is to be described so as to be towards A, as in the accompanying diagram. In each diagram A is the centre of similarity of the two squares.



Exercises.

308. Draw a quadrilateral ABCD from the data AB = 1, BC = 2, CD = 2·5, DA = 4, AC = 2 ; take any convenient point, O, in its plane and join OA, OB, OC, OD ; from a point a in OA draw ab parallel to AB to meet OB, from b draw bc parallel to BC to meet OC, and from c draw cd parallel to CD to meet OD. Prove that ad is parallel to AD ; measure the ratios OA : Oa, BC : bc, and show that they are equal.

309. In the preceding exercise construct the quadrilateral $abcd$ so that each side of it may be (i) $\frac{4}{5}$, (ii) $\frac{4}{3}$ of the corresponding side of ABCD.

310. The sides of a triangle are 1·3, 2, 2·1 in. : find by construction the sides of the inscribed squares having in succession one side lying on each side of the triangle.

* 311. The base of a triangle is a and the perpendicular to it from the vertex is p : prove that the side of the inscribed square standing on the base is $\frac{ap}{a+p}$ or $\frac{ap}{a-p}$, according as the opposite angular points of the square fall on the remaining sides of the triangle or on these produced.

312. The sides of a triangle are 3, 4, 5; a square is drawn with two sides lying on the sides 3 and 4 and the opposite vertex on the side 5. Find its side (i) when both sides fall within the sides of the triangle, (ii) when one of them falls on one side produced.

313. In an equilateral triangle of side 4 cm. inscribe a square: measure its side and also find it by calculation.

314. The sides of a triangle are 4, 5, 7: inscribe an equilateral triangle with one side parallel to the side 7.

315. In a triangle ABC inscribe an equilateral triangle so that two of its sides meeting in AB are equally inclined to AB.

316. In a sector of radius 15 mm. and angle 72^0, a square (with two angular points on the arc) and an equilateral triangle (with one angular point on the arc) are described: find their sides.

317. Draw a semicircle of radius ·6 in., and inscribe in it a square one side of which lies along its diameter: measure the side of the square.

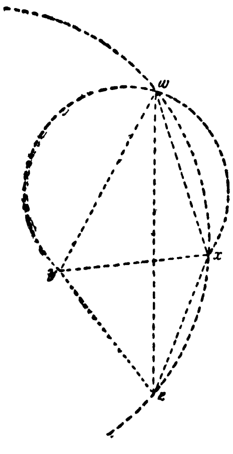
318. In the preceding exercise, if d = the diameter of the semicircle and x = the side of the square, prove that $d^2 = 5x^2$.

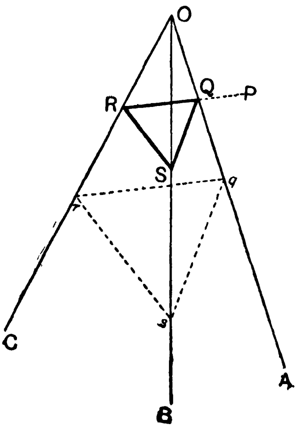
---

319. O is a fixed point and P any point on the perimeter of a given triangle ABC; OQ is drawn at a constant inclination to OP and such that the ratio OQ:OP is fixed. As P moves on the perimeter of ABC, prove that Q traces out a similar triangle.

320. Construct any equilateral triangle with its vertices on three given concurrent straight lines.

321. Solve the problem above when further one side of the equilateral triangle goes through a fixed point.





Solution. Let OA, OB, OC be the concurrent lines, P the given pt. Draw any equilateral $\Delta\ xyz$; on xy, xz describe segments of $\odot$s containing $\angle$s respectively equal to AOC, AOB [63], and let these intersect in w. On OA, OB, OC take Oq, Or, Os respectively equal to Ox, Oy, Oz. It is readily seen that the Δs Oqs and wxz, Oqr and wxy, are congruent : hence $qs=xz$, $qr=xy$, $\angle rqs=\angle yxz$, and $\therefore$ the Δqrs is equilateral.

Thus an equilateral Δ of any size may be placed with its vertices severally on the three given lines. Next, through P draw PQR parl. to qr, cutting OA, OC; and take $OS:Os=OQ:Oq$ or $OR:Or$. Then the Δ QRS is similar to qrs [XXXII, *Cor.*], and is $\therefore$ equilateral; and its side OR passes through P. There are two other solutions in which the side going through P is parl. to qs and rs successively.

322. Construct any square with three of its vertices on three concurrent straight lines.

323. Solve the problem above when a side of the square is of given length.

324. Inscribe a square in a given segment of a circle.

325. Prove that the point of contact of two touching circles is one of their centres of similitude, and find out whether it is the internal or external centre.

326. AC, CB are diameters of two circles touching at C, and AB is bisected in D; any circle described with D as centre meets one of the given circles in E_1, F_1, and the other in E_2, F_2. Prove that E_1, C, F_2 and E_2, C, F_1 are collinear.

327. Construct a triangle ABC, given the vertex A, and the points where BC touches the incircle and the excircle opposite to A.

328. Draw a circle to touch two given straight lines and to pass through a given point. [See *Prob.* vi, § 9; the solution there given may be shortened by employing centres of similarity.]

* 329. The vertices of a triangle are the direct centres of similitude of its incircle with its excircles; and the inverse centres of similitude of its excircles with each other.

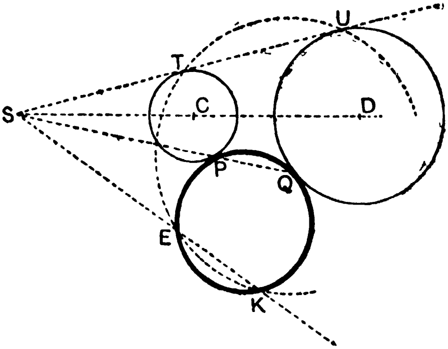
330. In the preceding exercise find the inverse centres of similitude of the incircle with the excircles, and the direct centres of the excircles with each other.

331. Find the internal and external centres of similitude of the circumcircle of a triangle and its nine-point circle. [See XIX, *Cors.* 1 and 2.]

332. Any secant through a centre of similitude of two circles divides each of them into pairs of similar segments.

333. A variable circle PQR touches two given circles at P and Q: prove that the straight line PQ always passes through a fixed point.

* 334. Apply the exercise above to draw a circle to touch two given circles and to pass through a given point. [See also Exer. 155.]



Solution. Join C, D the centres of the given ○s, and draw a common tangent TU meeting CD in S, a centre of similitude. Let E be the given pt.; draw the ○ ETU, and let SE meet this ○ again in K. Then the ○ passing through E and K and touching either circle (§9, *Prob.* iv) also touches the other circle.

Let the ○ through E and K be drawn to touch the ○ of centre C at P. Join SP, cutting this circle again at Q, and the ○ of centre D at Q′ if possible.

Then SP.SQ = SE.SK = ST.SU ;

but as S is a centre of similitude, ST.SU = SP.SQ′ [XXXIII, Note].

Hence SP.SQ′ = SP.SQ, *i.e.*, Q and Q′ coincide, and ∴ the ○ EKP touches the second ○ at Q.

335. A point A and two straight lines XOX′, YOY′ are given : construct an equilateral triangle having one vertex at A and the other two severally on XX′, YY′.

336. Solve the problem above when the given straight lines XX′, YY′ are parallel.

Miscellaneous Examples IV.

1. The ratio of the perimeters of two similar polygons equals the ratio of any pair of corresponding sides.

2. In the side BC of a triangle ABC a point D is taken such that $BD=\frac{1}{n}$ of BC; in AD a point E is taken such that $AE=\frac{1}{m}$ of AD. If BE is joined to meet AC in F, find the value of the ratio AF : AC.

3. Find by geometrical construction the side of a square whose area is 2·7 sq. in., and verify your result by calculation.

4. Two circles touch internally at A, and BC a chord of the outer touches the inner at T: prove that BT : TC = BA : AC.

5. If D is any point in the side BC of a triangle ABC, prove that the radii of the circumcircles of the triangles ABD, ACD are in the ratio AB : AC.

6. AD is drawn to touch the circumcircle of a triangle ABC and to meet BC produced: prove that the radii of the circumcircles of the triangles ABD, ACD are as AD to CD.

7. The vertex of a triangle standing on a given base moves on a fixed straight line . prove that the locus of its centroid is a straight line.

8. Find a point such that the perpendiculars from it to the three sides of a given triangle are proportional to three given lengths. [See Exer. 282.]

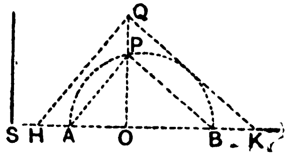
9. A fixed point O is at a distance of 1·5 in. from a given straight line AB; P is any point on AB, and Q is taken on OP so that rect. OP.OQ = 3·15 sq. in. Find the locus of Q.

*10. A fixed point O is in the plane of a given circle; OP is drawn to any point of the circumference, and Q is taken on it so that rect. OP.OQ = a given area. Prove that the locus of P is a circle.

11. Take a point O on a given circle, and plot any curve similar to the circle, O being the centre of similitude.

12. Through a given point draw a straight line meeting three given concurrent straight lines so that the two intercepts between the concurrent lines may be in a given ratio.

13. Describe a rectangle equal to a given square and having its sides in a given ratio.



Solution. Let S be a side of the given sq., and *l*:*m* the given ratio. Take any str. line AB and divide it at O so that AO:OB = *l*:*m* [XXIV]; on AB describe a semicircle, and erect OP perp. to AB. On OP (produced if necessary) take OQ = S; draw QH, QK parl. to PA, PB, to meet AB. Then HO, OK are the sides of the required rectangle.

$\because$ $\angle \mathrm{HQO} = \angle \mathrm{APO}$ and $\angle \mathrm{KQO} = \angle \mathrm{BPO}$,

$\therefore$ $\angle \mathrm{HQK} = \angle \mathrm{APB} =$ one rt. angle.

Hence rect. $\mathrm{HO}.\mathrm{OK} = \mathrm{QO}^2$ [XXX, *Cor.* 1] $= \mathrm{S}^2 =$ [*Cons.*].

Also $\frac{\mathrm{HO}}{\mathrm{AO}} = \frac{\mathrm{QO}}{\mathrm{PO}} = \frac{\mathrm{KO}}{\mathrm{BO}}$ [XXI, *Cor.* 1.];

$\therefore$ *alternando*, $\frac{\mathrm{HO}}{\mathrm{KO}} = \frac{\mathrm{AO}}{\mathrm{BO}} = \frac{l}{m}$ [*Cons.*];

$\therefore$ HO, OK are the sides of the rect. required.

14. From a point A outside a given circle draw a straight line APQ meeting the circle in two points P, Q, so that AP = PQ. When is the solution not possible?

15. A point A and a straight line BC are given in the plane of a fixed circle; draw a straight line APQ to cut BC in Q and the circle in P, so that AP = PQ, and find when the problem is impossible.

16. Prove that the area of a regular polygon of n sides inscribed in a circle is n times the area of the rectangle contained by the radius of the circle and one-fourth of the shortest diagonal of the polygon.

17. Describe a triangle similar to a given triangle, so as to have one of its vertices at a given point and the other two severally on two given straight lines. [See Exer. 335.]

18. Describe a semicircle to touch two sides of a given square and to have the ends of its diameter on the other two sides of the square. If a is one side of the square, prove that the radius of the semicircle is nearly $\frac{3}{5}a$.

*19. BAB′ is a given angle, and P and Q are points within it such that $\angle \mathrm{PAB} = \angle \mathrm{QAB'}$; if PM, PM′ and QN, QN′ are perpendicular respectively to AB, AB′, prove that rect. PM.QN = rect. PM′.QN′.

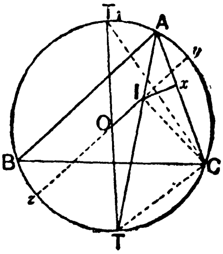
20. Prove the converse of the preceding theorem,—that if with the previous notation rect. PM.QN = rect. PM′.QN′, then AP, AQ are equally inclined to AB, AB′ respectively.

*21. ABC is a triangle right-angled at C; BC is bisected in M and BM in K; the straight line MI parallel to AC meets AB in N and the line through K parallel to the bisector of the angle BAC in I. Prove that IN = BN − IM.

22. The incircle of a triangle ABC touches the side BC in D, and the excircle opposite to the angle A touches it in D′: if DIX is a diameter of the incircle, prove that A, X, D′ are collinear.

23. The midpoint, A′, of the side BC of a triangle is joined to the incentre I; A′I is drawn to meet the perpendicular from A on BC in L. Prove that AL equals the radius of the incircle, and enunciate the corresponding result for the escribed circles.

24. With the notation of Prop. XVII and Exer. 167, prove that $AI.IT = 2Rr$, $OI^2 = R^2 - 2Rr$.



Solution. Join TC, T_1C, IC; draw Ix perp. to AC, and let OI meet the circum○ in y, z.

Now $\angle TIC = \angle IAC + \angle ICA = \frac{1}{2}(A+C)$;

and $\angle TCI = \angle TCB + \angle ICB = \angle TAB + \angle ICB = \frac{1}{2}(A+C)$:

$\therefore \quad \angle TIC = \angle TCI$, and $TC = TI$.

Also, $\because \angle TT_1C = IAx$, and $\angle TCT_1 = 90^0 = IxA$,

$\therefore \quad \triangle$s TCT_1, IAx are equiangular.

Hence $\frac{Ix}{IA} = \frac{TC}{TT_1}$; $\therefore \; AI.TC = TT_1\,Ix$, *i.e.*, $AI.IT = 2Rr$.

Again, as yz is bisected in O and divided at I,

$$\therefore zI.Iy = Oy^2 - OI^2;$$

$$\therefore OI^2 = Oy^2 - zI.Iy = Oy^2 - AI.IT = R^2 - 2Rr.$$

By a similar method we can show that $AI_1.I_1T = 2Rr_1$, and $OI_1^2 = R^2 + 2Rr_1$.

25. With the notation of § 10, Ch. IV, prove that $AI.AI_1 = bc$, $BI.BI_1 = ca$, $CI.CI_1 = ab$; $AI.II_1 = BI.II_2 = CI.II_3 = 4Rr$.

*26. With the same notation prove that

$$rr_1 = (s-b)(s-c), \quad r_2r_3 = s(s-a),$$

and $\quad rr_1r_2r_3 = \Delta^2$. [See Exer. 8.]

27. $ABDE$ is a quadrilateral having BD parallel to AE; through C, the point of intersection of AD and BE, FCG is drawn parallel to BD or AE and terminated by the other sides. Prove that $\frac{1}{AE} + \frac{1}{BD} = \frac{1}{CF}$ or $\frac{1}{CG}$.

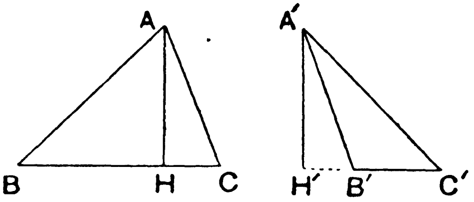
CHAPTER VII.

APPLICATION OF PROPORTION TO AREAS AND RECTANGLES.

§ 18. AREAS OF SIMILAR FIGURES.

Proposition XXXVI. Theorem.

Two triangles having equal heights have their areas proportional to the bases.



Let the Δs ABC, A′B′C′ have equal heights, AH, A′H′: to prove that area ABC : area A′B′C′ = BC : B′C′.

$\because$ area ABC $= \frac{1}{2}$ rect. AH.BC,

and area A′B′C′ $= \frac{1}{2}$ rect. A′H′.B′C′, [24]

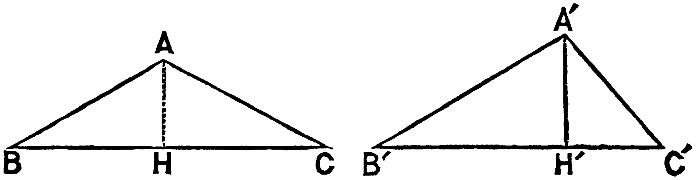
$$\therefore \quad \frac{\Delta\ ABC}{\Delta\ A'B'C'} = \frac{\frac{1}{2}AH.BC}{\frac{1}{2}A'H'.B'C'} = \frac{BC}{B'C'}, \text{ as } AH = A'H'.$$

Cor. 1. Triangles having equal bases have their areas proportional to the heights.

Cor. 2. Parallelograms having equal heights have their areas in the ratio of the bases; for they are the doubles of triangles with equal heights.

Proposition XXXVII. Theorem.

Two triangles having one angle of one equal to one angle of the other have their areas proportional to the rectangles contained by the sides about the equal angles.



Let the Δs ABC, A'B'C' have $\angle ABC = \angle A'B'C'$:
to prove that area ABC : area A'B'C' = rect. AB.BC : rect. A'B'.B'C'.

Draw AH, A'H' the altitudes of the Δs.

$\because$ $\angle ABH = \angle A'B'H'$ [*Hyp*], and $\angle AHB = \angle A'H'B'$ = rt.angle,

$\therefore$ Δs ABH, A'B'H' are equiangular,

and $\frac{AH}{A'H'} = \frac{AB}{A'B'}$. [XXVI]

Now
$$\frac{\Delta ABC}{\Delta A'B'C'} = \frac{\frac{1}{2}AH.BC}{\frac{1}{2}A'H'.B'C'} = \frac{AH}{A'H'} \cdot \frac{BC}{B'C'}$$
$$= \frac{AB}{A'B'} \cdot \frac{BC}{B'C'}. \quad [\textit{Proved}]$$

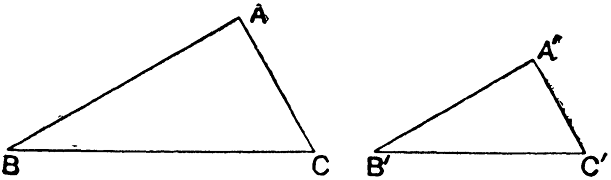
$\therefore$ $\Delta ABC : \Delta A'B'C' = AB.BC : A'B'.B'C'$.

Cor. 1. Two triangles are equivalent if one angle of one equals one angle of the other and the rectangles of the sides containing these angles are equal.

Cor. 2. Equiangular parallelograms have their areas in the ratio of the rectangles of the sides about the equal angles.

Proposition XXXVIII. Theorem.

Two similar triangles have their areas proportional to the squares on corresponding sides.



Let the Δs ABC, A′B′C′ be similar, the sides BC, B′C′ being homologous: to prove that area ABC : area A′B′C′ = $BC^2 : B'C'^2$.

$\because$ the Δs are similar, $\therefore$ $\frac{AB}{A'B'} = \frac{BC}{B'C'}$. [*Def.*]

But $\because$ the Δs have $\angle ABC = \angle A'B'C'$,

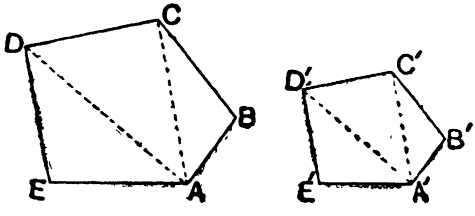
$$\therefore \frac{\Delta ABC}{\Delta A'B'C'} = \frac{AB.BC}{A'B'.B'C'} \quad \text{[XXXVII]}$$

$$= \frac{AB}{A'B'} \cdot \frac{BC}{B'C'} = \frac{BC}{B'C'} \cdot \frac{BC}{B'C'}. \quad [\textit{Proved}]$$

$$\therefore \quad \Delta ABC : \Delta A'B'C' = BC^2 : B'C'^2$$
$$= AB^2 : A'B'^2 = AC^2 : A'C'^2.$$

Proposition XXXIX. Theorem.

Two similar polygons have their areas proportional to the squares on corresponding sides.



Let ABCDE, A′B′C′D′E′ be similar polygons, the sides AB, A′B′ being homologous: to prove that area ABCDE: area A′B′C′D′E′ = $AB^2 : A'B'^2$.

Join A, A′ to the non-adjacent vertices of the polygons; then the Δs ABC, ACD, ADE are respectively similar to the Δs A′B′C′, A′C′D′, A′D′E′. [XXXIV, *Cor.* 2]

$\therefore\ \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB^2}{A'B'^2}$ [XXXVIII]; $\frac{\triangle\ ACD}{\triangle\ A'C'D'} = \frac{CD^2}{C'D'^2} = \frac{AB^2}{A'B'^2}$ [*Def.*];

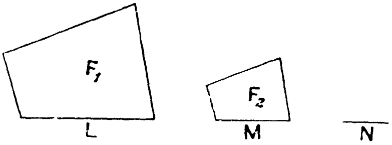
and $\frac{\triangle ADE}{\triangle A'D'E'} = \frac{DE^2}{D'E'^2} = \frac{AB^2}{A'B'^2}$;

hence $\frac{AB^2}{A'B'^2} = \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E'}$. [§ 12, vii]

Thus $\frac{\text{area ABCDE}}{\text{area A'B'C'D'E'}} = \frac{AB^2}{A'B'^2}$,

$= \frac{BC^2}{B'C'^2}, \frac{CD^2}{C'D'^2}$, *etc.*

Cor. If three straight lines L, M, N, are in continued proportion, any rectilineal figure described on the first is to the similar figure described on the second as L : N.



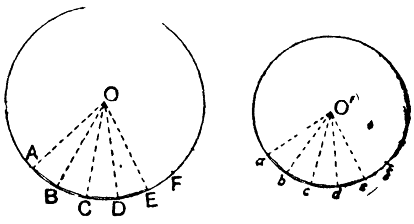
For, area of the first *fig* area of second $= L^2 : M^2$.
But $\because$ L : M = M : N, $\therefore$ $L^2 : M^2 = L . N$ [§ 12, vi]
Hence area of first *fig.* : area of second
= first str. line : third str. line.

The results of the two preceding propositions may be thus enunciated: similar rectilineal figures are to one another in the *duplicate ratio* of their corresponding sides.

It is seen from the second note to Prop. XXXII, that a circle is the limiting form of a regular polygon inscribed in it when the number of the sides of the polygon is indefinitely increased and the length of each side is indefinitely decreased.

Now, in the diagram here, polygon ABCDE... : polygon *abcde*... = $AB^2 : ab^2$, as regular polygons of the same number of sides are similar. But the $\triangle$s OAB, O'*ab* are evidently similar; hence AB : *ab* = OA : O'*a*.

$\therefore$ polygon ABCDE... : polygon *abcde*... = $OA^2 : O'a^2$.



Making the number of sides infinite, we see that in the limit ○ABCDEF ..: ○*abcdef* .. = $OA^2 : O'a^2$; *i.e., the areas of circles are proportional to the squares of their radii.*

Exercises.

337. The areas of two similar triangles are 192, 588 sq. ft. respectively, and one side of the former is 15ft.: find the length of the corresponding side of the latter.

338. Find the length of a straight line drawn parallel to the base of an equilateral triangle of side 1·8 cm. and cutting off at the vertex a triangle having $\frac{1}{6}$ of its area.

339. Prove that the area of the triangle formed by joining the midpoints of the medians of a triangle is $\frac{1}{16}$ of the area of the triangle.

340. Through the vertices A,B,C, of an equilateral triangle straight lines are drawn respectively perpendicular to the sides AB,BC,CA, so as to form another triangle: compare the areas of the two triangles.

341. The sides AB,AD, of a parallelogram ABCD, are bisected in E, F, and CE,CF are joined: find the ratio of the area of the triangle CEF to that of the parallelogram.

342. CD is drawn perpendicular to the hypotenuse AB of a right-angled triangle ABC: prove that $\triangle CAD : \triangle CBD = CA^2 : CB^2$.

343. Prove that similar triangles are in the duplicate ratio of the radii of their inscribed circles.

*344. Prove that similar cyclic polygons are to one another as the squares of the radii of their circumscribed circles.

345. Through the centroid of a triangle a straight line is drawn parallel to one side and terminated by the other two sides: compare the areas of the two parts into which the line divides the triangle.

346. Within a given triangle ABC find a point D such that the triangles ADB, BDC, CDA may be equal in area. If AD produced meets BC in X, find the ratio AX:DX.

347. Two triangles ABC,DBC, of equal area, are on the same side of the base BC; AC, BD meet in E and AB, CD in F. Prove that EF (produced if necessary) bisects both AD and BC.

348. Two straight lines ACD, BCE intersect in C, and AB, DE are joined: from the greater of the two triangles ABC, DEC, cut off a part equal to the less by a straight line drawn through C.

*349. A triangle ABC is right-angled at C, and CD is perpendicular to AB; on AB, AC the equilateral triangles ABX, ACY are described, and DX is drawn. Prove that the triangles ADX, ACY are equivalent.

Solution. ∵ CD is perp. to the hypotenuse,

∴ rect. AD.AB = sq. on AC. [XXX, *Cor.* 2]

∴ rect. AD.AX = rect. AC.AY [*Hyp.*], so that in the $\triangle$s ADX, ACY, $\angle$ DAX = $\angle$ CAY, and the rectangles of the containing sides are equal;

∴ these $\triangle$s are equivalent. [XXXVII, *Cor.* 1]

This is a particular case of the more general theorem given below.

*350. Similar and similarly described figures are drawn on the sides of a right-angled triangle: prove that the figure on the hypotenuse is equal to the sum of the figures on the sides about the right angle.

351. A triangle ABC is right-angled at C, and CD is perpendicular to AB; a polygon described on AB is equivalent to ABC. Prove that the similar polygon similarly described on AC is equivalent to ADC.

352. Draw a triangle with area equal to the difference of the areas of two given similar triangles and similar to each of them.

353. Draw a polygon with area equal to the sum of the areas of two given similar polygons and similar and similarly described to each of them.

354. Draw an equilateral triangle of side 1·5″ and cut off at one corner an equilateral triangle of side ·5″· describe an equilateral triangle equivalent to the remaining trapezium. Measure the side of this triangle and also find it by calculation.

355. In the diameter, PQ, of a circle produced a point O is taken; OT is a tangent to the circle, PN another tangent cutting OT in N, and C is the centre. Prove that $\triangle$ OPN : $\triangle$ OTC = OP:OQ.

356. To the circumcircle of a triangle ABC the tangent at A is drawn to meet BC in D: prove that DB:DC = AB^2:AC^2.

Solution. ∵ ∠ DAC = ∠ ABD [48], and ∠ s ADC, ADB are the same, ∴ $\triangle$s ADB, ADC are equiangular and similar.

Hence $\triangle$ ADB : $\triangle$ ADC = AB^2 : AC^2. [XXXVIII]

But $\triangle$ ADB : $\triangle$ ADC = DB : DC, for they have the same height;

∴ DB : DC = AB^2 : AC^2.

357. Two circles intersect at A and B; the tangents at A to each circle meet the other circle in X and Y. Prove that

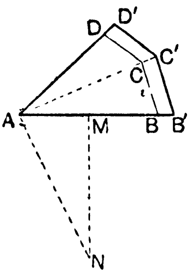
$$\triangle ABX : \triangle ABY = AB^2 + BX^2 : AB^2 + BY^2.$$

358. Construct any triangle with sides proportional to 12, 14, 17; bisect its area by a straight line parallel to the mean side, and measure the length of this line.

359. Construct a triangle ABC, given BC = 1·2″, ∠ B = 50°, ∠ C = 73°; divide it into three equal parts by straight lines parallel to BC.

*360. Bisect a given triangle by a straight line parallel to one of the sides.

*361. Describe a polygon similar to a given polygon and such that its area is to that of the given polygon as 5 to 4.



Solution. Let ABCD be the given polygon. Bisect AB in M, draw MN perp. to AB and equal to it; join AN, and mark off $AB' = AN$.

Draw B'C' parl. to BC to meet AC, and C'D' parl. to CD to meet AD.

The polygon AB'C'D' is evidently similar to ABCD.

Also, $AB'^2 = AN^2 = AM^2 + MN^2 = AM^2 + AB^2 = 5AM^2$, and $AB^2 = 4AM^2$.

$$\text{Hence } \frac{\text{fig. } AB'C'D'}{\text{fig } ABCD} = \frac{AB'^2}{AB^2} = \frac{5}{4}.$$

The side AB' is $\sqrt{\frac{5}{4}}$, if AB is the unit: thus the process amounts to finding the surd $\frac{1}{2}\sqrt{5}$. In the preceding exercise the surd $\frac{1}{2}\sqrt{2}$ is required. For the construction of such surds see the Note to Prop. V.

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362. Bisect the area of a given triangle by a straight line (i) perpendicular to a given side, (ii) parallel to a given straight line.

363. Two regular hexagons are respectively inscribed in and described about the same circle: compare their areas.

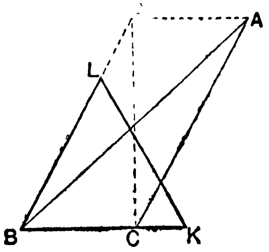
364. The middle points of the sides of a regular octagon are joined successively to form another regular octagon: compare the areas of the two figures.

365. Prove that the area of a regular octagon inscribed in a circle is the mean proportional between the areas of the inscribed and circumscribed squares.

366. Construct an isosceles triangle equivalent to a given triangle and having its vertical angle equal to one angle of the given triangle.

367. Construct a right-angled triangle equivalent to a given triangle and having one angle equal to one angle of the given triangle.

*368. Construct an equilateral triangle equivalent to a given triangle.



Solution. Let ABC be the given Δ Draw AD parl. to BC, and at B make the ∠ CBD = 60° by a line meeting AD. Join CD; between BC, BD take a mean proportional [XXXI], and on these lines mark off BK, BL equal to it.

Then the Δ BKL is the required triangle.

∵ BK = BL and ∠ KBL = 60°, ∴ each of the ∠s BKL, BLK is 60°, and the Δ BKL is equilateral.

Again in the Δs DBC, BKL, the ∠ at B is common, and rect. BC.BD = BK^2 = BK.BL, ∴ the Δs are equivalent. [XXXVII, *Cor. 1*]

But ΔBDC = ΔABC [24, *a*]; ∴ the Δ BKL is equivalent to the Δ ABC.

369. Draw a parallelogram with adjacent sides of lengths 32 and 27 mm. and included angle of 70°: construct a rhombus of equal area having also an angle of 70°, and measure its side.

370. Draw a parallelogram of area 3·6 sq. in having adjacent sides in the ratio 3:5 and included angle of 75°.

371. Construct a triangle of area 5·8 sq. in. with sides proportional to 2, 3, 4.

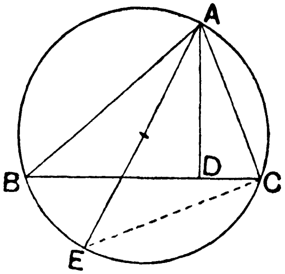
372. Divide a given square into three parts of equal area by straight lines parallel to one of the diagonals.

*373. Construct a triangle similar to a given triangle and equal in area to another given triangle.

§ 19. RECTANGLES IN CONNECTION WITH PROPORTION.

Proposition XL. Theorem.

In any triangle the rectangle contained by two of the sides is equal to the rectangle contained by the diameter of the circumcircle of the triangle and the perpendicular drawn from the point of intersection of the two sides to the third side.



Let ABC be a △ ; draw AD perp to BC ; circumscribe the ○ ABEC and draw its diameter AE: to prove that rect. AB.AC = rect. AD.AE. Join EC.

∵ ∠ AEC = ∠ ABD, being angles in the same segment, and ∠ ACE = ∠ ADB, being rt. angles,

∴ the △s AEC, ABD are equiangular.

Hence $\frac{AB}{AE} = \frac{AD}{AC}$; [XXVI]

∴ rect. AB.AC = rect. AD.AE.

Cor. The numerical value of the product of the three sides of a triangle is equal to that of twice the product of its area and the diameter of the circumcircle.

∵ AB.AC = AD.AE, multiplying by the numerical value of BC, we get AB.AC.BC = BC.AD.AE.

But BC.AD represents twice the area of the triangle ;

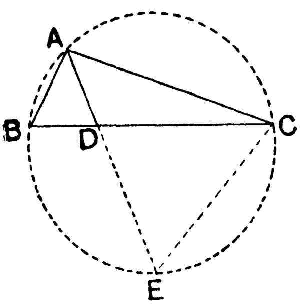
∴ AB.BC.CA = 2 (area) × diameter of circum○.

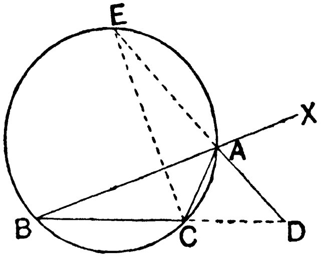
With the usual notation, this result may be written thus —

$$abc = 2\,\Delta d \text{ or } 4\,\Delta R.$$

Proposition XLI. Theorem.

In any triangle the rectangle contained by two of the sides is equal to the square on the bisector of the angle contained by them terminated by the third side, together with the rectangle of the segments of this side made by the bisector.





Let ABC be a $\triangle$; draw AD bisecting the $\angle$ BAC, and meeting the side BC: to prove that rect. AB.AC = AD^2 + rect. BD.DC.

Draw the circum ○ of the triangle, produce AD to meet it in E, and join EC.

$\because$ $\angle$ AEC = $\angle$ ABD, and $\angle$ EAC = $\angle$ BAD [*Hyp.*],

$\therefore$ the $\triangle$s AEC, ABD are equiangular.

Hence $\frac{AB}{AE} = \frac{AD}{AC}$; $\therefore$ rect. AB.AC = rect. AD.AE.

But rect. AD.AE = AD. (AD + DE)

= AD^2 + AD.DE

= AD^2 + BD.DC. [VIII, *Cor.* 1]

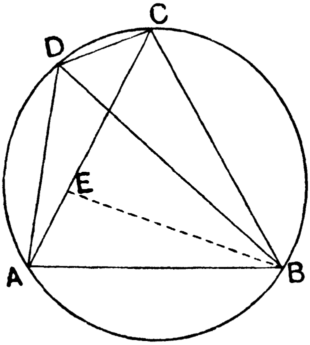
$\therefore$ rect. AB.AC = AD^2 + rect. BD.DC.

Cor. If AD bisects the exterior angle at A and meets BC produced (as in the second diagram), it can be similarly shewn that rect. AB.AC = rect. BD.DC − AD^2.

The results of the Prop. and its Cor. may be included under one statement thus : the square on the bisector (terminated by the base) of the internal or external vertical angle of a triangle is equal to the difference of the rectangles contained by the sides of the triangle and by the segments of the base made by the bisector.

Proposition XLII. Theorem.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle equals the sum of the rectangles contained by the pairs of opposite sides.



Let ABCD be a cyclic quad. ; to prove that
rect. AC BD = rect. AB.CD + rect. AD.BC.

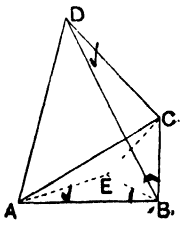
At B draw BE meeting AC and making ∠ ABE = ∠ DBC, so that ∠ ABD = ∠ EBC.

In the Δs ABE, DBC, ∠ ABE = ∠ DBC, ∠ BAE = ∠ BDC [45];
∴ these Δs are similar, so that $\frac{AB}{BD} = \frac{AE}{DC}$; [XXVI]
∴ rect. AB.DC = rect. AE.BD .... (i)

In the Δs ABD, EBC, ∠ ABD = ∠ EBC, ∠ ADB = ∠ ECB [45];
∴ these Δs are similar, so that $\frac{AD}{EC} = \frac{BD}{BC}$. [XXVII]
∴ rect. AD.BC = rect. EC.BD .... (ii)

Hence rect. AC.BD = rect. (AE + EC).BD
= rect. AE.BD + rect. EC.BD
= rect. AB.DC + rect. AD.BC, from (i) and (ii).

Cor. If the quadrilateral is not cyclic, the rectangle contained by the diagonals is less than the sum of the rectangles contained by the pairs of opposite sides.



Make ∠ ABE = ∠ DBC, and, as ∠ BAC is not equal to ∠ BDC, make also ∠ BAE = ∠ BDC. Join CE.

Then the Δs ABE, DBC are similar, and rect. AB.DC = rect. AE.BD, as before.

Also from these $\triangle$s $\frac{AB}{BD}=\frac{EB}{BC}$, and $\angle ABD = \angle EBC$;

$\therefore$ the $\triangle$s ABD, EBC are similar [XXVIII], and rect. AD.BC = rect. EC.BD, as before.

Hence rect. AB.DC + rect. AD.BC = rect. (AE + EC). BD, which is > rect. AC.BD, $\because$ AE + EC is > AC.

Exercises.

374. Draw a triangle of sides 12, 25, 30 cm., bisect its smallest angle, and measure the length of the bisector and the segments of the opposite side made by it. Hence verify Prop. XLI.

375. In the exercise above find the diameter of the circumcircle of the triangle, and hence verify Prop XL.

376. ABCD is a cyclic quadrilateral, and AC, BD meet in E: prove that AB.BC : CD.DA = BE:ED

377. The areas of two triangles inscribed in the same circle are to one another as the continued products of their sides.

378. Construct a triangle, given the base, the vertical angle and the rectangle contained by the sides.

379. ABC is a triangle; from A two straight lines AD, AE are drawn equally inclined to AB, AC, the former meeting the base and the latter the circumcircle of the triangle. Prove that

rect. AD. AE = rect. AB. AC.

380. A circle is described about an equilateral triangle ABC, and P is any point on its arc BC: prove that PA = PB + PC.

381. ABCD is a cyclic quadrilateral, and rect. AB.BC = rect. CD.DA: prove that AC and BD intersect at a point which bisects one of them.

382. The diagonals of a cyclic quadrilateral are at right angles: prove that the sum of the rectangles contained by the opposite sides is equal to twice the area of the figure.

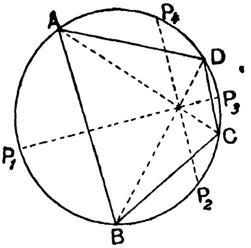
383. In a cyclic quadrilateral ABCD, BD bisects the angle at B: prove that AB + BC : BD = AC : CD.

384. ABCD is a cyclic quadrilateral, and rect. AB.CD = rect. AD.BC: prove that the tangents to the circumcircle at A and C meet on BD produced.

*385. Construct a triangle, given the lengths of two sides and the length of the perpendicular on the third side from the opposite vertex.

386. The bisector of the vertical angle of a triangle terminated by the circumcircle is bisected by the base of the triangle: prove that the square on either side is twice the square on the adjacent segment of the base.

*387. ABCD is a cyclic quadrilateral : find a point P on the circumcircle such that (i) rect. PA.PC = rect. PB.PD, (ii) rect. PA.PD = rect. PB.PC.



Solution. Joining AC, BD, it is seen that rect. PA.PC = rect. of diameter of ○ and perp. from P on AC [XL]; so for rect. PB.PD. Hence the perps. from P on AC and BD are equal, and ∴ P lies on one of the bisectors of the angles between these lines [42]. Hence the construction: let AC, BD intersect in O; bisect the angles at O by lines meeting the ○ce in P_1, P_2, P_3, P_4; these are the positions of the pt. required.

In the second case the angle between AD and BC produced has to be bisected ; there are only two solutions.

*388. Construct a triangle, given the lengths of two sides and the length of the bisector of the included angle terminated by the base.

389. ABCDE is a regular pentagon and AC, BD meet in O: prove that (i) $2AC=(\sqrt{5}+1)AB$, (ii) AC is divided at O in extreme and mean ratio.

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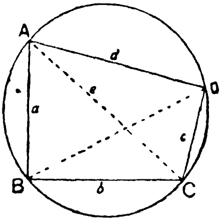
*390. The lengths of the sides AB, BC, CD, DA, of a cyclic quadrilateral are a, b, c, d, respectively: prove that

$$AC^2=\frac{(ac+bd)(ad+bc)}{ab+cd},\quad BD^2=\frac{(ac+bd)(ab+cd)}{ad+bc}.$$



Solution. Let $AC=e$, $BD=f$; and let R be the radius of the ○. From the △ ABC, $abe=4R(\triangle ABC)$, [XL, *Cor.*];
so also, $cde=4R(\triangle ADC)$.
By addition, $e(ab+cd)=4R$ (Quad.).
By similar reasoning, $f(ad+bc)=4R$ (Quad.).

Hence, $e(ab+cd)=f(ad+bc)$, and $e^2(ab+cd)=ef(ad+bc)$.
But $ef=ac+bd$ [XLII];
∴ $e^2(ab+cd)=(ac+bd)(ad+bc)$.
So also f^2 may be found.
It is seen that $e:f=ad+bc:ab+cd$.

391. AB is a fixed chord of a circle, Q the midpoint of the minor arc AB, and P any point on the major arc : prove that PA+PB varies as PQ.

392. A′ is the midpoint of the base of a triangle ABC, and AA′ meets the circumcircle in K : prove that rect. AB.BK = rect. AC.CK.

Miscellaneous Examples V.

1. The radii of two circles are 5 to 6, and the area of the second is 25.2 sq. cm.: find the area of the first.

2. A semicircle is described about a right-angled triangle, and on the two sides about the right angle two other semicircles are drawn outside the triangle: show that the areas of the crescents formed between the first semicircle and the other two are together equal to the area of the triangle.

3. Transform a given triangle into an equivalent isosceles right-angled triangle.

4. In a given triangle inscribe a rhombus with an angle of 45°, so that one side may fall on one side of the triangle and the opposite vertices lie on the other two sides severally.

5. In a triangle the squares on two unequal sides are in the ratio of their projections on the third side: prove that the triangle is right-angled.

*6. A point O within a triangle ABC is joined to the vertices, and OA, OB, OC meet the opposite sides of the triangle in K, L, M respectively: prove that $\frac{OK}{AK}+\frac{OL}{BL}+\frac{OM}{CM}=1$. Investigate the result when O is outside the triangle.

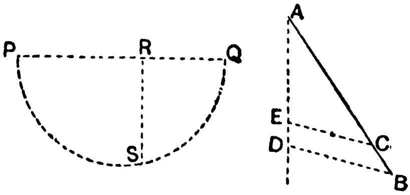
7. ABC, AD are triangles of equal area with the angle at A common: if BC, DE meet at F, prove that AF bisects both BE and CD.

8. Construct two squares or two equilateral triangles, whose areas shall be in a given ratio and whose sum shall be equal to a given square or a given equilateral triangle.

9. Construct two straight lines of given difference, so that the rectangle contained by them shall be equal to a given square.

*10. Divide a given straight line into two parts such that one part is to the line as $\sqrt{2}$ is to $\sqrt{3}$.

Solution. Let AB be the given line. Take any str. line PRQ, mark off PR = 3, RQ = 2 on any convenient scale, and construct the mean proportional RS to PR and RQ.



Through A draw any convenient line, mark off AD = PR, AE = RS, join DB, and draw EC parl. to DB.

$$\because \frac{PR}{RS}=\frac{RS}{RQ}, \quad \therefore \frac{PR^2}{RS^2}=\frac{PR}{RS}\cdot\frac{RS}{RQ}=\frac{3}{2}; \quad \text{thus } \frac{PR}{RS}=\frac{\sqrt{3}}{\sqrt{2}}.$$

Hence $\frac{AC}{AB}=\frac{AE}{AD}$ [XXVII] $=\frac{SR}{RP}$ [*Cons.*] $=\frac{\sqrt{2}}{\sqrt{3}}$.

11. ABC is a triangle: two circles are drawn, one touching AB at A and passing through C, the other touching AC at A and passing hrough B. If the common chord of these circles meets BC in D, prove that BD:DC $= AB^2 . AC^2$.

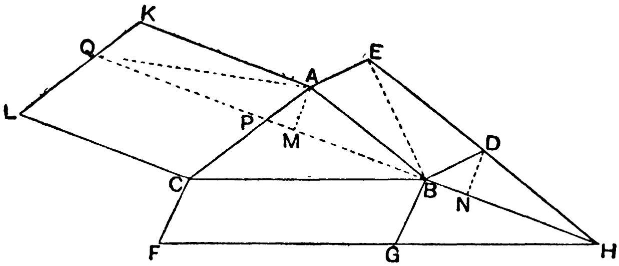
12. ABC is a triangle inscribed in a circle; AD and AE are drawn parallel to the tangents to the circle at B and C respectively o meet BC. Prove that BD: CE $= AB^2 : AC^2$.

13. Draw a semicircle in a given sector of a circle so that the wo arcs touch one another and the ends of the diameter of the semicircle lie on the radii of the sector

14. The sides AB, AC, of a triangle are 4″ and 5″ long, and he bisector of the angle A is $3\frac{1}{4}$ in.: find the segments of the base made by the bisector, drawing the triangle to any convenient scale.

15. E, F, G, H are the midpoints of the sides AB, BC, CD, DA, of a square, and AF, BG, CH, DE form by their intersections another square PQRS : if *a*, *b*, *c*, *d* are the midpoints of PQ, QR, RS, SP, compare the areas of the figures *abcd*, ABCD.

*16. On the sides AB, BC, of a triangle ABC the parallelograms ABDE, BCFG are outwardly described; DE, FG are produced to meet in H, and on CA a parallelogram is described whose second side is equal and parallel to BH. Prove that the area of this parallelogram is equal to the sum of the areas AD and BF.



Solution. Let CAKL be the paralgm. described on CA, and let HB meet AC, KL in P, Q Join BE, AQ; draw AM, DN perp. to HQ.

∵ the Δs APQ, DBH are on equal bases HB, PQ,

$$\therefore \frac{\Delta\,APQ}{\Delta\,DBH} = \frac{AM}{DN}.$$

Now the Δs ABM, DHN are equiangular,

$$\therefore \frac{AM}{DN} = \frac{AB}{DH}\ [\text{XXVII}] = \frac{DE}{DH}\ [16]; \quad \therefore \frac{\Delta\,APQ}{\Delta\,DBH} = \frac{DE}{DH}.$$

$$\text{But } \frac{\Delta\,DBE}{\Delta\,DBH} = \frac{DE}{DH}\ [\text{XXXVII}], \quad \therefore \frac{\Delta\,APQ}{\Delta\,DBH} = \frac{\Delta\,DBE}{\Delta\,DBH}.$$

Thus Δ APQ = Δ DBE, and ∴ fig. AQ = fig. AD.

Similarly, fig. CQ = fig. BF; hence the paralgm. CAKL is the sum of the paralgms AD and BF

17. C is any point in a straight line AB; on AC, CB similar right-angled triangles ACE, CFB are similarly described, E and F being the right angles. Prove that $AC.CB = AE.CF + CE.BF$.

18. D, E, F are points on the base BC of a triangle ABC such that AF bisects the angle A, and AD, AE trisect it: prove that $AB.DF.EC = AC.EF.DB$.

19. P, Q, R are points in the sides BC, CA, AB respectively, of a triangle ABC, such that $BP\ PC = CQ\ QA = AR\ RB$, and RQ meets BC in S. prove that $BS:SC = CP^2\ BP^2$.

20. AA', BB', CC' are three parallel line-segments such that ABC, $A'B'C'$ are straight lines; AB', $A'C$ meet in P and AC', $A'B$ in Q. Prove that PQ is parallel to BB'.

21. Given of a triangle the base in magnitude and position, and that the rectangle of the sides varies as the area of the triangle: prove that the locus of the vertex is a circle.

22. On the base BC of a triangle ABC find a point P such that (rect. BD DC + sq. on AD) may be equal to a given square.

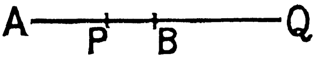
23. From any vertex of a triangle ABC the perpendicular is drawn to the opposite side; from F, the foot of this perpendicular, and FQ are drawn perpendiculars to the other sides. Prove that PQ is a fixed length.

CHAPTER VIII.

MISCELLANEOUS PROPOSITIONS INVOLVING PROPORTION.

§ 20. HARMONICAL SECTION.

In Prop. XXIV a method has been given by means of which a finite line may be divided, internally or externally, in a given ratio. When it is thus divided in the *same ratio numerically*, the line is said to be *harmonically divided* (§ 13, *Def.* and *Note*).



If AB is the finite straight line, P and Q the points of internal and external section in the same ratio, the four points A, B, P, Q are said to form a *harmonic range*, which is briefly denoted by the symbol (AB, PQ) The ends of the line and the points of section are shown separately, and P and Q are said *to be harmonically conjugate* to one another *with regard to* A and B, or *to separate* A and B *harmonically*.

If P and Q separate A, B harmonically, A and B separate P, Q harmonically also. For, by definition $\frac{AP}{PB}=\frac{AQ}{QB}$ numerically; hence $\frac{PB}{PA}=\frac{BQ}{AQ}$ (*invertendo*), and $\frac{PB}{BQ}=\frac{PA}{AQ}$ (*alternando*), so that B and A are harmonically conjugate points with regard to P and Q, or PQ is harmonically divided at A and B.

The reason for calling such section of the line harmonical is that the *lengths* AP, AB, AQ form a *harmonical progression* as defined in Algebra. For,

$$\because \frac{AP}{PB}=\frac{AQ}{QB}, \quad \therefore \frac{QB}{AQ}=\frac{PB}{AP}; \quad \therefore \frac{AQ-AB}{AQ}=\frac{AB-AP}{AP},$$

or $1-\frac{AB}{AQ}=\frac{AB}{AP}-1$, and $2=\frac{AB}{AQ}+\frac{AB}{AP}$; $\therefore \frac{2}{AB}=\frac{1}{AP}+\frac{1}{AQ}$,

so that AB is the harmonical mean between AP and AQ.

Again, $\frac{AQ}{AP}=\frac{BQ}{BP}=\frac{AQ-AB}{AB-AP}$, so that the third length is to the first length as (the third – the second) is to (the second – the first).

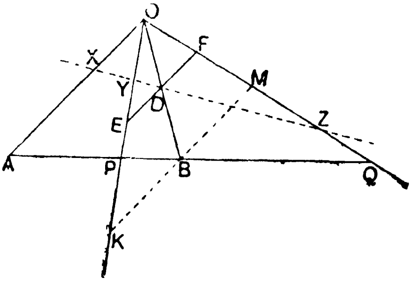
If any point O is joined to the four points A, B, P, Q, of a harmonic range, the figure thus formed is called a *harmonic pencil*; O is said to be the *vertex* of the pencil, and OA, OB, OP, OQ are called its *rays*. The pencil is denoted by the symbol O(AB, PQ), the rays OP, OQ being *harmonically conjugate* with regard to OA, OB.

The following results give the fundamental properties of a harmonic pencil.

(i) Any straight line parallel to one ray of a harmonic pencil is divided into equal parts by the other three rays.

Let EDF parl. to OA meet the other three rays of the harmonic pencil O (AB, PQ): then DE = DF.

Draw KBM parl. to EDF.



By similar Δs, $\frac{ED}{BK}=\frac{DO}{BO}=\frac{DF}{BM}$; $\therefore\ \frac{ED}{DF}=\frac{BK}{BM}$.

Again, $\frac{AO}{BK}=\frac{AP}{PB}$, and $\frac{AO}{BM}=\frac{AQ}{BQ}$, by similar Δs;

but $\frac{AP}{PB}=\frac{AQ}{BQ}$, as (AB, PQ) is harmonic.

$\therefore\ \frac{AO}{BK}=\frac{AO}{BM}$, *i.e.*, BK = BM; and $\therefore$ ED = DF.

(ii) Any straight line meeting all the rays of a harmonic pencil is divided harmonically at its points of intersection with the rays

Let the str. line XYDZ meet the rays of the harmonic pencil O(AB, PQ): then it is harmonically divided. Draw EDF parl. to the ray AB of the pencil; then ED = DF, as proved above.

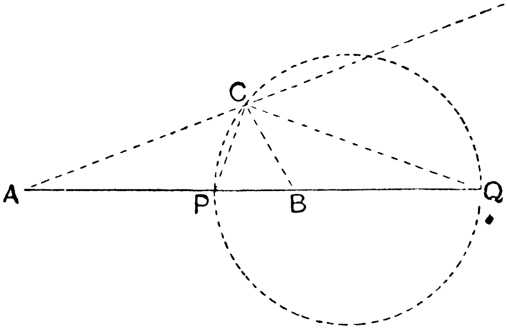
Now $\frac{XY}{YD}=\frac{XO}{DE}$, from the similar Δs XYO, DYE;

and $\frac{XZ}{ZD}=\frac{XO}{DF}$, from the Δs XZO, DZF.

But $\frac{XO}{DE}=\frac{XO}{DF}$; $\therefore\ \frac{XY}{YD}=\frac{XZ}{ZD}$,
so that (XD, YZ) is a harmonic range.

Proposition XLIII. Theorem.

The locus of a point whose distances from two fixed points are in a constant ratio is a circle.



Let A, B be the fixed pts., $l : m$ a constant ratio.

Let C be any pt. such that CA : CB $= l : m$ to prove that C moves on a fixed circle.

Divide AB internally and externally at P and Q in the ratio $l : m$ [XXIV], then P, Q are fixed pts. [XX]. Join CA, CB, CP, CQ.

∵ $\frac{CA}{CB} = \frac{l}{m} = \frac{AP}{PB}$, ∴ CP bisects ∠ ACB [XXIII],

so also, ∵ $\frac{CA}{CB} = \frac{AQ}{QB}$ [*Cons.*], ∴ CQ bisects ∠ ACB externally.

Hence the ∠ PCQ is right; and C lies on the ○ described on the fixed line PQ as diameter [46, *a*].

The same is true of any other pt. satisfying the condition.

Conversely, if (AB, PQ) is the constructed harmonic range and the circle on PQ as diameter is described, it can be shewn that every point on this circle has its distances from A and B in the constant ratio.

∴ the locus of the pt. C is a fixed circle.

Exercises.

393. In the figure of the Proposition take a point not on the circumference of the circle PCQ, and prove that it does not satisfy the condition of the theorem.

394. (AB, PQ) is a harmonic range and the circle on PQ as diameter is described: prove directly that every point on the circumference has its distances from A and B in a constant ratio.

395. The base of a triangle is 3 cm. in length and its vertical angle contains $52\frac{1}{2}^\circ$: construct the triangle so that the ratio of its sides may be 5:4.

396. Draw to a suitable scale a triangle ABC with BC = 1·8′, CA = 1·5′, AB = 1′, and find a point whose distances from **A, B, C** are respectively as 3, 4, 5.

397. Construct a triangle, given the base, the ratio of the sides, and (i) the altitude, (ii) the length of the external bisector of the vertical angle.

398. AB is a chord of a given circle: find a point on the circumference whose distances from A and B are in a given ratio.

399. ABCD is a trapezium with AB and CD parallel, and AC, BD meet in O: prove that the straight line through O parallel to AB or CD and terminated by BC and AD, is the harmonic mean between AB and CD. (See Misc. Ex. IV, 27.)

400 If a straight line parallel to one of the rays of a pencil is divided into equal segments by the other three rays, then the pencil is harmonic.

401. If (AB, PQ) is a harmonic range, and O is the midpoint of AB, prove that OP, OB, OQ are in geometrical progression.

*402 If O is the midpoint of a finite straight line AB and P and Q are taken in AB and AB produced so that rect. $OP.OQ = OB^2$, prove that (AB, PQ) is a harmonic range.

403. ABC is a straight line: find two points in it, H and K, equidistant from C and such that (AB, HK) is a harmonic range.

404. From a point P outside a circle (centre O) the tangents PS, PT are drawn; PO meets ST in Q and the circle in A and B. Prove that (AB, PQ) is a harmonic range.

405. Given three points of a harmonic range or three rays of a harmonical pencil, show how to find the remaining point or ray.

406. In the preceding exercise if one of the points is midway between the other two, what is the position of its conjugate point? (See Exer. 228.)

407. If (AB, PQ) is harmonic, prove that

$$\tfrac{1}{2} \text{ rect. } AB.PQ = \text{ rect. } AQ.BP = \text{ rect. } AP.BQ.$$

408. If (AB, PQ) is harmonic, and O is the midpoint of AB, prove that

(i) rect. AQ.QB = rect. OQ.QP,
(ii) rect. AP.PB = rect OP.PQ,
and (iii) $OP:OQ = AP^2:AQ^2$.

409. From a point P outside a circle the tangents PS, PT are drawn; any straight line through P meets the chord ST in Q and the circle in A and B. Prove that the range (PQ, AB) is harmonic.

Solution. Let C be the centre of the $\odot$; then the $\triangle$s PTC, PSC are congruent, and ∴ the $\triangle$s PTN, PSN also.

Hence $\angle$ PNT = rt. angle; also $\angle$ PTC is right.

∴ $PT^2 = PN.PC$ [XXX, *Cor. 2*]; and $PT^2 = PA{\cdot}PB$ [IX].

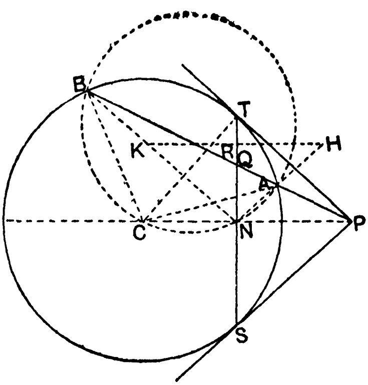
Thus PA.PB = PN PC, and A, B, C, N are concyclic [VIII, *Cor. 2*].

∴ $\angle$ ANP = $\angle$ ABC = $\angle$ BAC = $\angle$ BNC.

But $\angle$ TNP = $\angle$ TNC;

∴ the remainders, $\angle$ ANT and $\angle$ BNT, are equal, and TN bisects the $\angle$ ANB.

Also CNP, being perp. to TNS, bisects the suplt. of the $\angle$ ANB.



∴ PA:PB = AN:NB = AQ:QB [XXII]; *i. e.*, (PQ, AB) is harmonic.

410. In a harmonic pencil if one ray bisects the angle between two other rays, it is perpendicular to its conjugate ray. Conversely, if two conjugate rays are perpendicular to one another, they bisect the internal and external angles between the other two rays.

Solution (of Converse). In the fig. of the preceding Exer., suppose N (PQ, AB) a harmonic pencil with two conjugate rays NP, NQ at rt. angles. From *any* pt. R in NQ draw KRH parl. to NP; then KR = RH [Theorem i, § 20]. Also $\angle$ KRN = $\angle$ RNP = 90° = $\angle$ HRN; hence the $\triangle$s HRN, KRN are congruent, and $\angle$ KNR = $\angle$ HNR. Thus NQR bisects the $\angle$ ANB and ∴ CNP bisects the angle exterior to it.

411. The diagonals of a parallelogram and the straight lines through their point of intersection parallel to the sides of the parallelogram, form a harmonic pencil.

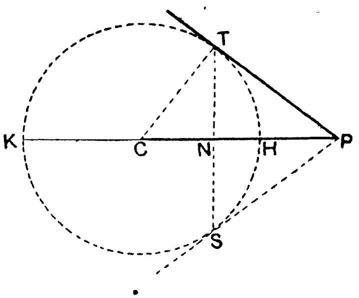
✓ 412. Construct the arithmetical, geometrical and harmonical means between two given lengths.

Solution. Draw any str. line PC, and mark off PH, PK, the two lengths. Bisect HK in C, and on HK as diameter draw a $\odot$; draw PT, PS tangents, and join TS, cutting PHK in N.

Then it is easily shown that (PN, HK) is a harmonic range;
$\therefore$ PN is the H.M. between PH and PK [§ 20].

Also $\because$ $PT^2 = PH.PK$ [IX],
$\therefore$ PT is the mean proportional, *i e.* G.M. between PH, PK.

Again $PK - PC = CK = CH = PC - PH$;
$\therefore$ PC is the A.M. between PH, PK.



413. A, B, C, D being four points in a straight line, find two points, H, K, in the line so that the ranges (AB, HK), (CD, HK), are both harmonic.

414. If a range (AB, PQ) is harmonic, then the circle on AB as diameter cuts orthogonally every circle through P and Q; and that on PQ as diameter cuts orthogonally every circle through A and B.

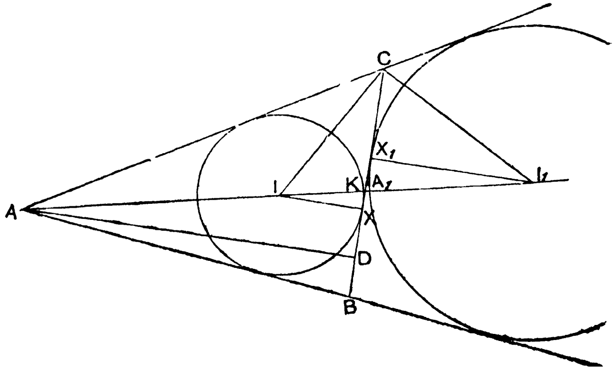
415. If two circles cut orthogonally, any diameter of one of them is cut harmonically at the points where it meets the other.

416. A, P, B, Q are four collinear points: if the circle on AB as diameter cuts orthogonally a circle through P and Q, then the range (AB, PQ) is harmonic.

417. The incircle of a triangle ABC touches the sides BC, CA, AB in X, Y, Z respectively, and YZ produced meets BC in X′: prove that the range (BC, XX′) is harmonic, and state the corresponding result for an excircle.

418. In a triangle ABC, H is the orthocentre, O the circumcentre, I, I_1, I_2, I_3 are the centres of the four circles which touch the three sides; and AI meets BC in K. Prove that the pencils A (HO, II_2), A(HO, II_3), and the range (AK, II_1), are all harmonic.

419. With the notation of the preceding exercise, if AD, IX, I_1X_1 are perpendiculars to BC, and A_1 is the midpoint of BC, prove that (DK, XX_1) is a harmonic range and that $A_1X^2 = A_1K.A_1D$.



Solution. $\because$ CI, CI$_1$ bisect the angles at C,
$\therefore$ AI : IK = AC : CK = AI$_1$: I$_1$ K;
thus the range (AK, II$_1$) is harmonic.
But AI : IK = DX : XK, and AI$_1$: I$_1$ K = DX$_1$: X$_1$K; [XXI]
$\therefore$ the range (DK, XX$_1$) is also harmonic.
Now BX = $s-b$, and CX$_1$ = $s-b$; [*Note*, Prop. XVII]
and BA$_1$ = A$_1$C. $\therefore$ A$_1$X = A$_1$X$_1$.

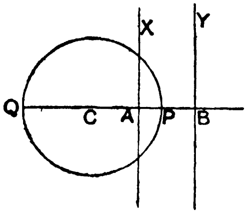
Thus, as $\dfrac{DX}{XK}=\dfrac{DX_1}{X_1K}$, $\therefore$ $\dfrac{KX_1}{KX}=\dfrac{DX_1}{DX}$;

$$\therefore \frac{KX_1+KX}{KX_1-KX}=\frac{DX_1+DX}{DX_1-DX}, \textit{ i.e.}, \frac{2A_1X}{2A_1K}=\frac{2A_1D}{2A_1X}.$$

Hence $A_1X^2 = A_1K.A_1D$; see Exer. 401.

§ 21. POLE AND POLAR.

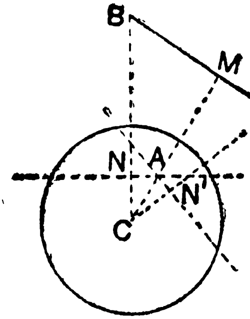
Def. If on any radius of a given circle two points are taken, one within and the other without the circle, so that the rectangle contained by their distances from the centre is equal to the square on the radius; then a straight line through *either* point at rt. angles to the radius is called the *polar of the other point*, and *this point* with regard to the polar is called *its pole*. Thus if C is the centre and r the radius of the circle, and $CA.CB = r^2$, and AX, BY are perpendicular to CAB, BY is called the *polar of* A, and A is the *pole of* BY. So also B and AX are mutually pole and polar.



Two points like A and B above are said to be each the *inverse* of the other; the given circle is called the *circle of inversion*, and its radius the *radius of inversion*. If AB meets the circle in P and Q, it is readily proved that (AB, PQ) is a harmonic range. (See Exer. 402.)

Proposition XLIV. Theorem.

If a point lies on the polar of another, the second point lies on the polar of the first.



Let the pt. B lie on the polar of A with regard to a ⊙ of centre C and radius $= r$: to prove that A lies on the polar of B with regard to the ⊙.

Let CA meet the polar of A in M; then $\angle$ CMB = rt. angle [*Def.*].

Draw AN perp. to CB.

∵ $\angle$ BMA + $\angle$ ANB = 2 rt. angles,

∴ pts. A, N, B, M are concyclic. [47]

Hence CN.CB = CA.CM [VIII, *Cor.* 1] $= r^2$. [*Hyp.*]

Also NA is perp. to CB;

∴ NA is the polar of B [*Def.*], *i. e.*, A lies on the polar of B.

Cor. 1. If B′ is another pt. on the polar of A, it can be similarly proved that AN′, the perp. from A to CB′, is the polar of B′; and so also for other pts. on the line MB. Hence, *if any number of points* B, B′,..., *be collinear, their polars are concurrent,* the point of concurrence, A, being *the pole of the line on which they lie.*

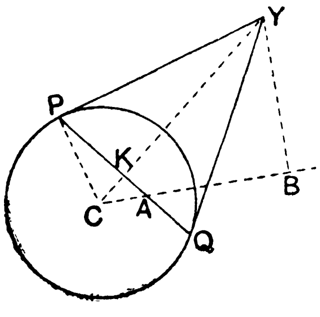
Cor. 2. Conversely, it is seen that *if any number of straight lines be concurrent, their poles are collinear,* the line of collinearity being *the polar of the point at which they meet.* Thus in the fig., if AN, AN′, AN″....are all concurrent, their poles B, B′, B″...all lie on MB, the polar of A.

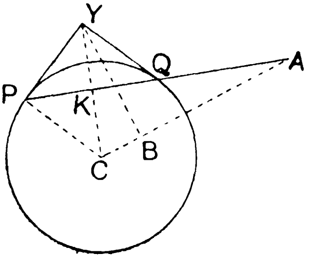
Cor. 3. If we take MB in the fig. as the line joining B,B′, we see that its pole is at the intersection of AN, AN′; thus *the line joining any two points is the polar of the point of intersection of the polars of the two points.* Conversely, *the point of intersection of any two lines is the pole of the line joining the poles of the two lines.*

Def. Two points, as A and B in the fig., such that the polar of either passes through the other, are said to be *conjugate points* with regard to the circle; and two lines, as AN and BM, such that the pole of either lies on the other, are said to be *conjugate straight lines.*

Proposition XLV. Theorem.

The locus of the intersection of tangents, drawn to a circle at the extremities of chords which all pass through a given point, is the polar of the point.





Let C be the centre and r = the radius of the given ⊙, and A the given pt.; let PY, QY be tangents at the extremities of any chord PQ passing through A: to prove that the locus of Y is the polar of A with regard to the ⊙.

Join CY, cutting PQ in K; join CP, and draw YB perp. to CA, produced if necessary.

$\because$ PY = QY, $\angle$ PYK = $\angle$ QYK [42], and YK is common to the $\triangle$s YPK, YQK, $\therefore$ these $\triangle$s are congruent.

$\therefore$ $\angle$ YKP = YKQ = rt. angle, and $\therefore$ $\angle$ PKC = rt. angle.

Also ∠ CPY = rt. angle ; [41]

∴ $CP^2 = CK.CY$. [XXX, *Cor.* 2]

Again, ∵ ∠ YKA = rt. angle = ∠ YBA,

∴ Y, K, A, B are concyclic [45 or 47] ;

and ∴ CB.CA = CK.CY [VIII, *Cor.* 1] = $CP^2 = r^2$.

Further BY is at rt. ∠s to CAB ;

∴ BY is the polar of A. [*Def.*]

So also any other pt. of intersection may be shown to lie on the polar of A.

Cor 1. Conversely, it can be proved in the same way that *if from points in a given straight line tangents are drawn to a given circle, the chords of contact all pass through a fixed point,* which is the *pole of the given line.*

Cor 2. If from an external point tangents are drawn to a circle, the chord of contact produced is the polar of the given point.

In the first fig. of the Prop., all points of BY form the locus; in the second fig., the points of BY within the circle form no part of the locus, as the tangents cannot intersect within the circle. When the given point is on the circle, the polar is the tangent at the point.

Def. When each side of a triangle is the polar of the opposite vertex with regard to a circle, the triangle is said to be *self-conjugate* or *self-polar* with regard to the circle, and the circle is said to be the *polar circle* of the triangle.

Exercises.

420. Prove that the point of intersection of two straight lines is the pole of the straight line joining their poles.

421. Prove the corollaries of Prop. XLV independently.

422. Find the locus of the poles of all straight lines passing through a given point.

423. Draw a circle of radius 12 mm. and take a point P at a distance of 21 mm. from its centre: construct the polar of P. If this polar meets the circle in S and T, shew that PS and PT are tangents to the circle.

424. Draw a circle of diameter 1·5″ and take a straight line at a distance of 1·2″ from its centre: take any three points on the line and draw from them the three pairs of tangents to the circle. Verify that the chords of contact are concurrent, and find the position of their point of meeting.

425. What is the polar of the centre of a circle with regard to the circle? And what is the pole of any given diameter?

426. Given a point and a straight line: construct any circle for which they are pole and polar respectively.

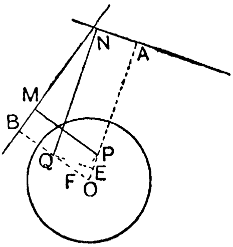
427. Given two concentric circles: find the locus of the poles of the tangents of one with regard to the other.

428. With regard to a circle the points A,B,C are the poles of the sides of the triangle PQR: prove that P,Q,R are the poles of the sides of the triangle ABC.

429. The angle between the polars of two points is equal to the angle subtended by the line joining them at the centre of the circle of reference.

430. If PM, QN be drawn perpendicular to the polars of Q,P respectively with regard to a circle whose centre is O, prove that PM : QN = OP : OQ.

Solution. Let AN,BM be the polars of P,Q; PM,QN perp. to BM,AN; and O the centre. If r = radius of the ○, OP.OA = OQ.OB = r^2 [*Def.*].
Draw PF, QE perp. to OQ,OP.
The pts. Q, F, E, P are concyclic [45];
∴ OE.OP = OF.OQ [VIII].
But OP.OA = OQ.OB;
∴ OP (OA − OE) = OQ(OB − OF),
i. e., OP.QN = OQ.PM.
Hence PM : QN = OP : OQ.



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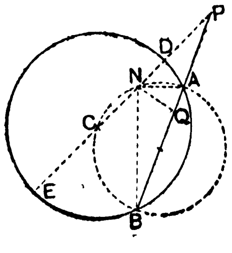
*431. Any chord of a circle passing through a fixed point is cut harmonically by the fixed point and its polar with regard to the circle. [See Exer. 409.]

432. On chords of a circle passing through a fixed point the points harmonically conjugate to the point are taken: prove that their locus is the polar of the fixed point.

Solution. Let C be the centre of the given ⊙ and P the fixed pt.; draw any chord, PAB, of the ⊙ and take Q in it so that (AB, QP) is harmonic: it is required to find the locus of Q.



Draw the diameter PDE, and let the ○ about CAB cut it in N.

$\because PN \cdot PC = PA \cdot PB = PD.PE,$

and CP, PD, PE are given,

∴ PN is fixed, and N is a fixed pt.

Further $PD.PE = CP^2 - CD^2$,

and $PN.PC = CP^2 - CN.CP$,

∴ $CN.CP = CD^2$, and N and P are inverse pts.

Now the radii CA, CB are equal,

∴ ∠ CNB = ∠ CAB = ∠ CBA = ∠ PNA [47],

so that NP bisects the angle adjacent to ANB.

But N (AB,QP) is harmonic;

∴ NQ bisects the ∠ ANB, and ∴ ∠ PNQ = rt. angle.

Similarly, the harmonic conjugate of P on any other chord is shown to lie on the str. line through N perp. to CP, *i.e.*, on the polar of the pt. P. (See Exers. 409, 410.)

433. From a point A outside a given circle the chords APQ, ARS are drawn equally inclined to the diameter through A: prove that PS, QR intersect in a fixed point.

434. Prove the following construction for drawing the polar of a point P with regard to a circle having its centre at C:—draw any chord, AB, of the circle, as also the diameter DE, passing through P; let AE, BD meet in X, and AD, BE in Y; then XY is the polar of P.

435. If two circles cut orthogonally, the extremities of any diameter of either are conjugate points with regard to the other circle.

436. If two conjugate points with regard to a circle are taken, the circle described on the line joining them as diameter cuts the given circle orthogonally.

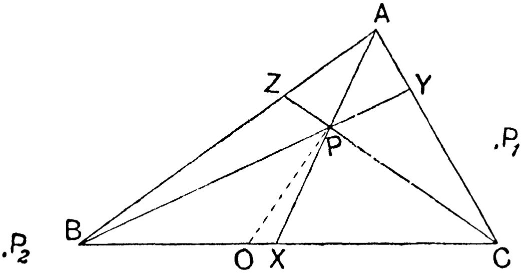
437. The square of the distance between two points conjugate with regard to a circle, is equal to the sum of the squares of the tangents drawn from the points to the circle. Prove this result, and examine the case when one of the points is inside the circle.

§ 22. GENERAL THEOREMS ON CONCURRENCY AND COLLINEARITY.

Proposition XLVI. Theorem.

If the straight lines joining any point to the vertices of a triangle, A, B, C, *meet the opposite sides of the triangle in the points* X, Y, Z, *respectively, then* BX.CY.AZ=XC.YA.ZB.

Conversely, if X, Y, Z *are points in the sides* BC, CA, AB *respectively such that* XB.CY.AZ = XC.YA.ZB, *then the straight lines* AX, BY, CZ *are concurrent.*



(i) Let any pt. P be joined to the vertices of the Δ ABC, and let AP, BP, CP meet the opposite sides in X, Y, Z : to prove that BO.CY.AZ = XC.YA.ZB.

∵ the Δs BPX, XPC have the same height, $\frac{BX}{XC} = \frac{\Delta BPX}{\Delta PCX}$;

so also $\frac{BX}{XC} = \frac{\Delta BAX}{\Delta XAC}$. [XXXVI]

$$\therefore \frac{BX}{CX} = \frac{\Delta BAX - \Delta BPX}{\Delta XAC - \Delta XPC} \text{[vii, § 12]} = \frac{\Delta BAP}{\Delta PAC}.$$

Similarly it can be shown that

$\frac{CY}{YA} = \frac{\Delta CPB}{\Delta PBA}$ and $\frac{AZ}{ZB} = \frac{\Delta APC}{\Delta PCB}$.

∴ by multiplication, $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$,

i. e., BX.CY.AZ = CX.YA.ZB.

(ii) Let X, Y, Z be pts. in the sides BC, CA, AB, such that BX.CY.AZ = XC.YA.ZB : to prove that AX, BY, CZ are concurrent.

Join BY, CZ, meeting in P ; let AP meet BC in O, if possible.

∵ the lines AP, BP, CP meet the opposite sides in O, Y, Z, we have BO.CY.AZ = OC.YA.ZB. [*Proved*]

But BX.CY.AZ = XC.YA.ZB ; [*Hyp.*]

$\therefore\ \frac{\text{BO}}{\text{BX}} = \frac{\text{OC}}{\text{XC}}$, or $\frac{\text{BO}}{\text{OC}} = \frac{\text{BX}}{\text{XC}}$. [*Alternando*]

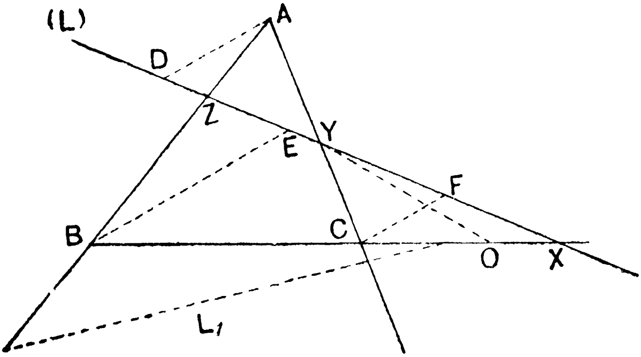
Thus BC is divided at O and X in the same ratio, which is impossible. [XX]

∴ O and X coincide, and AP, BP, CP are concurrent.

With the convention of signs adopted in § 4, it will be found that the two products have always the same sign, or that $\frac{\text{BX}}{\text{XC}} \cdot \frac{\text{CY}}{\text{XA}} \cdot \frac{\text{AZ}}{\text{ZB}} = +1$. Thus all the segments are positive in the given figure ; but if P is taken at P_1, XC and AZ will both be negative, and the product of the ratios will still be $+1$. So also other positions of P (as P_2) may be considered. It will be found that the sections of the sides are either *all internal* or *one internal and two external.*

Proposition XLVII. Theorem.

If a straight line intersects the sides BC, CA, AB, *of a triangle in* X, Y, Z, *respectively, then* BX.CY.AZ = CX.AY.BZ. *Conversely, if* X, Y, Z *are points in the sides* BC, CA, AB *respectively, such that* BX.CY.AZ = CX.AY.BZ, *then the points are collinear.*



(i) Let any str. line (L) cut the sides of the △ABC at X, Y, Z: to prove that BX.CY.AZ = CX.AY.BZ.

Draw any parl. lines through A, B, C, to meet (L) at D, E, F, respectively.

∵ the △s BXE, CXF are similar, ∴ $\frac{BX}{CX}=\frac{BE}{CF}$; [XXVI]

so also $\frac{CY}{AY}=\frac{CF}{AD}$, $\frac{AZ}{BZ}=\frac{AD}{BE}$.

∴ by multiplication, $\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=1$,

i. e., BX.CY.AZ = CX.AY.BZ.

(ii) Let X, Y, Z be pts. in the sides BC, CA, AB, such that BX.CY.AZ = CX.AY.BZ: to prove that X, Y, Z are collinear.

Join YZ, and let it meet BC in O, if possible.

∵ ZYO is a str. line,

we have BO.CY.AZ = CO.AY.BZ. [*Proved*]

But BX.CY.AZ = CX.AY.BZ; [*Hyp.*]

∴ $\frac{BO}{BX}=\frac{CO}{CX}$, or $\frac{BO}{CO}=\frac{BX}{CX}$. [*Alternando*]

Thus BC is divided at O and X in the same ratio, which is impossible. [XX]

∴ O and X coincide, and X, Y, Z are collinear.

With the usual convention of signs, it will be found that the products have always the same sign, or $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = +1$. Thus, in the figure given AY and BZ are negative, the other segments are all positive.

If the line is taken as L_1, the signs will be different, but the product of the ratios will still be $+1$. It will be found that the sections of the sides are either *all external* or *one external and two internal.* It should be noticed that in the enunciations of XLVI and XLVII the right hand sides are *differently written.*

Exercises.

438. Draw a triangle ABC with $BC = 28$, $CA = 15$, $AB = 19$; in BC, CA take BL, CM each $= 10$, and let AL, BM meet in O. If CO and AB meet in N, calculate AN and NB, and verify by measuring their lengths.

439. The straight lines joining the vertices A, B, C, of a triangle to any point O meet the opposite sides in D, E, F, respectively: if $\frac{BD}{DC} = \frac{1}{3}$, $\frac{CE}{EA} = \frac{4}{3}$, find $\frac{AF}{FB}$, and prove that $AO = 3\ OD$.

440. Draw the figure of the preceding exercise when E divides CA externally, and shew that the results still hold good.

441. In the figure of Prop. XLVI, join ZX, ZY, and prove that the pencil Z (BC, X Y) is harmonic.

Prove also that $\frac{AP}{AX} + \frac{BP}{BX} + \frac{CP}{CX} = 2$.

442. X, Y, Z are points on the sides BC, CA, AB respectively of a triangle, and Z (BC, XY) is a harmonic pencil: prove that AX, BY, CZ are concurrent.

443. By means of Prop. XLVI or XLVII construct a ratio which shall be the product or quotient of two given ratios, $a : b$ and $c : d$.

444. In the sides CA, AB, of a triangle ABC points M, N, are taken so that $CM : MA = AN : NB = a : b$; MN and BC meet in L. Prove that $BL : LC = a^2 : b^2$.

445. With the figure of Prop. XLVI prove that $\frac{AY}{YC} + \frac{AZ}{ZB} = \frac{AP}{PX}$; and hence construct a ratio which shall be the sum or difference of two given ratios.

446. If Y, Z are taken in AC, AB respectively so that $\frac{CY}{YA} = \frac{BZ}{AZ}$ (algebraically), and BY and CZ meet in P, prove that AP is parallel to BC.

447. Apply Prop. XLVI to prove the concurrence of the straight lines joining the vertices of a triangle to (i) the midpoints of the opposite sides; (ii) the points of contact of the incircle with the opposite sides; (iii) the points of contact of any excircle with the opposite sides; and (iv) the points of contact of the three excircles severally with the opposite sides.

448. Shew that the perpendiculars of a triangle are concurrent; and that two external bisectors and the third internal bisector of the angles of the triangle are also concurrent.

449. The three external bisectors of the angles of a triangle meet the respectively opposite sides in collinear points.

450. If a triangle is inscribed in a circle, the tangents at its vertices meet the respectively opposite sides in collinear points.

451. The incircle or any excircle of a triangle touches the sides BC, CA, AB in D, E, F respectively; BC and EF meet in X, CA and FD in Y, AB and DE in Z. Prove that XYZ is a straight line.

452. A straight line cuts the sides AB, BC, CD, DA of a quadrilateral at L, M, N, O respectively: prove that the numerical product of the ratios $\frac{AL}{LB}$, $\frac{BM}{MC}$, $\frac{CN}{ND}$, $\frac{DO}{OA}$, is unity. Extend the result to a polygon of any number of sides.

453. In the sides AB, BC, CD, DA, of a quadrilateral points L, M, N, O, are respectively taken so that AL.BM.CN.DO = BL.CM.DN.AO, prove that AC, LM, NO are concurrent.

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454. If from any point in the diagonal, AC, of a quadrilateral ABCD, two straight lines are drawn, one meeting AB,BC in X,Y respectively, and the other CD, DA in X′, Y′ respectively, then AX.BY.CX′.DY′ = XB.YC.X′D.Y′A.

455. The incircle of a triangle ABC touches BC, CA in X, Y respectively, and the excircle opposite to the vertex A touches AB in Z: if XYZ is a straight line, prove that the ∠ ABC is right.

456. A circle cuts the three sides BC, CA, AB, of a triangle respectively at D_1 and D_2, E_1 and E_2, F_1 and F_2: if AD_1, BE_1, CF_1 are concurrent, so also are AD_2, BE_2, CF_2.

457. If X and X′, Y and Y′, Z and Z′ are points taken respectively in the sides BC, CA, AB, of a triangle, so that XYZ, X′Y′Z′ are straight lines, prove that the points of intersection of

(i) BC and YZ′, CA and ZX′, AB and XY′,

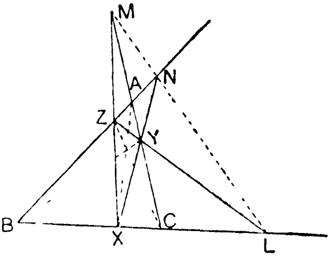
and (ii) BC and Y′Z, CA and Z′X, AB and X′Y,

are collinear.

*458. If X, Y, Z are points on the sides, BC, CA, AB, of a triangle such that AX, BY, CZ are concurrent, prove that YZ, ZX, XY meet the respectively opposite sides in collinear points.

Solution. Let YZ, ZX, XY meet BC, CA, AB in L, M, N, respectively.

∵ LYZ is a str. line,
∴ BL.CY.AZ = CL AY.BZ [XLVII].
But AX, BX, CZ are concurrent;
∴ BX.CY.AZ = XC.YA.ZB. [XLVI].
Hence, by division, $\frac{BL}{BX} = \frac{CL}{CX}$,
i.e., $\frac{BL}{CL} = \frac{BX}{CX}$.
Similarly it can be shewn that
$\frac{CM}{AM} = \frac{CY}{YA}$ and $\frac{AN}{B} = \frac{AZ}{ZB}$.



Hence by multiplication, $\frac{BL.CM.AN}{CL.AM.BN} = \frac{BX.CY.AZ}{XC.YA.ZB} = 1$;

∴ BL.CM.AN = CL.AM.BN, and L, M, N are collinear [XLVII].

It is seen that BC is divided harmonically at X and L. Taking the quadrilateral BCYZ, it is found that the opposite sides intersect in A and L, and the diagonals in the point (O) of concurrence; so that the lines joining A to O and A to L divide the side BC harmonically. (See Exer. 461 below.)

459. If a straight line, LMN, cuts the sides, BC, CA, AB, of a triangle at L, M, N, respectively, and if AL, BM, CN form a triangle, the lines joining A, B, C to the corresponding vertices of this triangle are concurrent.

If O is the point of concurrence, O and LMN are respectively called *pole* and *polar with regard to the triangle* ABC of the two preceding exercises.

460. If straight lines AP, BP, CP cut at X, Y, Z the sides, BC, CA, AB, respectively of a triangle, and if X′, Y′, Z′ are harmonically conjugate to X, Y, Z with regard to BC, CA, AB respectively, prove that

(i) X′, Y′, Z′ ; X, Y, Z′, are collinear points,

and (ii) AX, BY′, CZ′; AX′, BY′, CZ, are concurrent straight lines.

461. ABCD is a quadrilateral ; AB, CD meet in E and AD, BC in F. If AC, BD meet in O, then the pencil E (BC, OF) is harmonic.

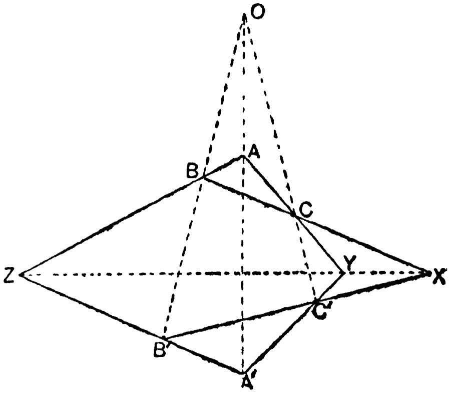
462. With the notation of the preceding exercise, if BD, EF meet in P and AC, EF in Q, then the ranges (FE, PQ), (BD, OP), (CA, OQ) are harmonic.

The figure formed by the four straight lines EAB, EDC, FDA, FCB is called a *complete quadrilateral*. The six intersections of these four lines, A, B, C, D, E, F, are called the *vertices* of the figure ; and the three lines AC, BD, EF, joining pairs of opposite vertices, are called its *diagonals*. If EO meets BC in F′, (BC, F′F) is a harmonic range as shown above.

*463. Prove that the midpoints of the three diagonals of a complete quadrilateral are collinear.

464. Pairs of points D_1 and D_2, E_1 and E_2, F_1 and F_2 are taken respectively in BC, CA, AB, the sides of a triangle, so as to be equidistant from the midpoints of these sides : if AD_1, BE_1, CF_1 are concurrent, so also are AD_2, BE_2, CF_2.

*465. If two triangles, ABC, A′B′C′, are such that AA′, BB′, CC′ are concurrent, prove that the pairs of sides BC, B′C′ ; CA, C′A′ ; AB, A′B′, intersect in collinear points.



Solution. Let AA′, BB′, CC′ meet in O, and the corresponding pairs of sides cut in X, Y, Z respectively.

∵ B′C′X crosses the sides of the △OBC,

$$\therefore\ \frac{BX}{XC}\cdot\frac{CC'}{C'O}\cdot\frac{OB'}{B'B}=1$$

(numerically);

$$\text{so also}\ \frac{CY}{YA}\cdot\frac{AA'}{A'O}\cdot\frac{OC'}{C'C}=1,\ \text{and}\ \frac{AZ}{ZB}\cdot\frac{BB'}{B'O}\cdot\frac{OA'}{A'A}=1.$$

$$\text{Hence, by multiplication,}\ \frac{BX.CY.AZ}{XC.YA.ZB}=1\ ;$$

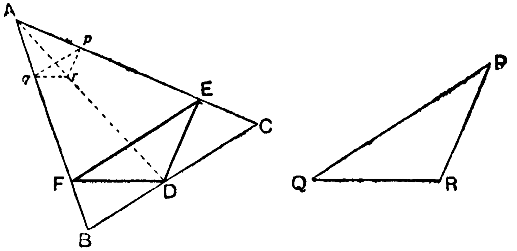
∴ from the △ABC, X, Y, Z are collinear.

466. If the corresponding sides of two triangles intersect in collinear points, the lines joining their corresponding vertices are concurrent.

The triangles in the two preceding exercises are said to be *in perspective* ; the point of concurrence for corresponding vertices is called their *centre of perspective*, and the line of collinearity for corresponding sides is called their *axis of perspective*.

§ 23. MISCELLANEOUS CONSTRUCTIONS.

(i) *In a given triangle to inscribe a triangle similar and similarly placed to a second given triangle.*



Let ABC be the first Δ, PQR the Δ to which the inscribed Δ is to be similar.

Across the ∠ BAC draw the str. line pq parl. to PQ; make ∠ qpr = ∠ QPR, ∠ pqr = ∠ PQR. Join Ar, and let it meet BC in D; draw DE, DF parl. to rp, rq to meet the other sides of the Δ ABC. Then DEF is the required Δ.

Evidently ∠ EDF = ∠ prq; also $\frac{DE}{pr} = \frac{AD}{Ar} = \frac{DF}{qr}$,

so that the Δs DEF, pqr are similar. [XXVIII]

But the Δs PQR, pqr are equiangular; [*Cons.*]

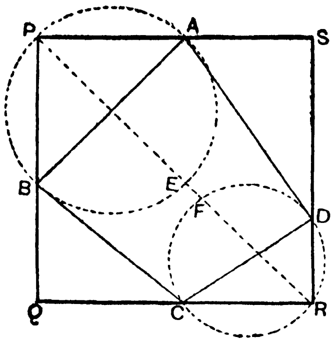
∴ the Δs DEF, PQR are similar. [XXVI]

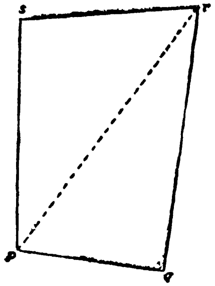
Also the sides of DEF are respectively parl. to those of pqr, and ∴ to those of PQR;

∴ the Δs DEF, PQR are similarly placed.

We can also draw qr or rp across the ∠ BAC so as to be parallel to QR or RP, and then complete the Δpqr so as to be similarly placed to PQR. Thus two more solutions may be obtained, and some of the points may fall on the sides of ABC produced. No new solutions are obtained by drawing the parallels across the ∠ ABC or ACB.

(ii) *To describe a square about a given quadrilateral, i.e., a square the sides of which pass severally through the vertices of the quadrilateral.*





Let ABCD be the quad.; on AB, CD as diameters describe ○s, and let E, F be the midpoints of the arcs of these ○s within the quad. Join EF, and let it meet the ○s in P, Q; join PB, PA, RC, RD, forming the quad. PQRS.

∵ ∠ BPA = 90° in a semi-○, and arc BE = arc EA,

∴ ∠ EPB = 45°; so also ∠ FRC = 45°.

∴ the △ PQR is an isosceles rt.-angled △.

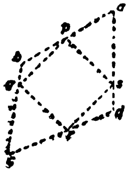
Similarly the △ PSR is an isosceles rt.-angled △, and ∴ PQRS is a square.

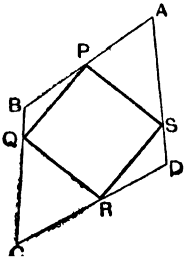
If the quad. to be circumscribed is not a square, but is similar to a second given quad. *pqrs*, the following method may be adopted.

Join *pr*; make ∠ ABE = ∠ *rps*, ∠ BAE = ∠ *rpq*, ∠ CDF = ∠ *prq*, ∠ DCF = ∠ *prs*. Join EF, and let it meet the ○s about the △s ABE, CDF in P, R. Join PA, PB, RC, RD, to form PQRS. The △s PQR, *pqr* will now be similar, and so also PSR, *psr*; hence the quad. PQRS is similar to the quad. *pqrs*.

Some of the sides of PQRS may pass when produced through the points A, B, C, D.

(iii) *To inscribe a square in a given quadrilateral, i.e., a square the vertices of which lie severally on the sides of the quadrilateral.*





Let ABCD be the quad. Take any square *pqrs*, and about it describe the quad. *abcd* similar to the quad. ABCD, as explained above. Divide the sides of the given quad.,

so that $AP:PB=ap:pb$, $BQ:QC=bq:qc$,
$CR:RD=cr:rd$, $DS:SA=ds:sa$.

Then it is seen that the figure PQRS is similar to *pqrs*, and is therefore a square.

Some of the points P, Q, R, S, may lie on the sides produced of ABCD.

Exercises.

467. Prove the following construction for describing a square about a given quadrilateral ABCD. Draw AO perpendicular and equal to BD; then OC is the direction of the side of the square through the vertex C. Examine the case when AC is itself perpendicular to BD.

468. Describe a square about a triangle ABC, so that one vertex is at A and the opposite sides pass through B and C severally.

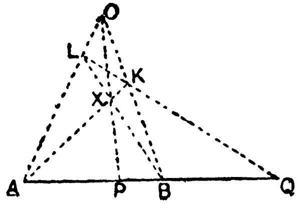
469. Describe a square so that (i) two vertices may lie on a given circle and the opposite side may touch the circle, (ii) two sides may touch the circle and the opposite vertex may lie on it.

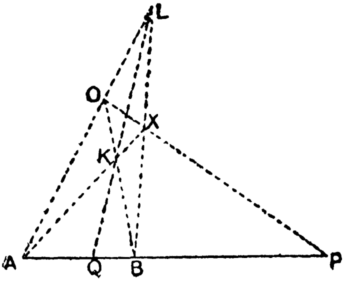
470. Construct a triangle similar to a given triangle so that the vertices may lie severally on three given concentric circles.

471. Given a circle and a point: construct by means of the ruler only the polar of the point with regard to the circle. Hence draw tangents to the given circle from any external point.

✓ *472. Find the harmonic conjugate of a point on a straight line with regard to the extremities of the line.

Solution. If A, B are the extremities of the line, and P the pt. on it, the harmonic conjugate of P may be found as in [XXIV] by drawing parallel straight lines. This requires the use of the *compasses* or the set square; the following method depends on the *straight ruler* only.





Join any pt. O to A, B, P, and on OP take a pt. X; let AX, BX meet OB, OA respectively at K, L; join KL, meeting AB in Q.

$\because$ LKQ crosses the sides of the Δ AOB, $\therefore \frac{AQ}{QB} \cdot \frac{BK}{KO} \cdot \frac{OL}{LA} = 1$.

But $\because$ AK, BL, OP concur at X, $\therefore \frac{AP}{PB} \cdot \frac{BK}{KO} \cdot \frac{OL}{LA} = 1$.

Hence $\frac{AQ}{QB} = \frac{AP}{PB}$, and Q is harmonically conjugate to P with regard to A, B.

Miscellaneous Examples VI.

1. Two points, P and Q, are equidistant from the centre of a given circle: if any circle through them cuts the given circle, prove that the common chord of the two circles is parallel to PQ.

2. Divide a given straight line, AB, internally or externally at C, so that $AC^2 - BC^2$ may be equal to a given area.

3. Prove that the geometrical mean of two given lengths is also the geometrical mean of their arithmetical and harmonical means. (See Exer. 412.)

4. D, E, F are the midpoints of the sides BC, CA, AB, of a triangle; EH parallel to AD meets FD produced in H, and BH, BE are joined. Prove that D is the centroid of the triangle BEH.

5. The base of a triangle is given in magnitude and position and the vertical angle is known: prove that the centroid and the nine-point centre move on circles.

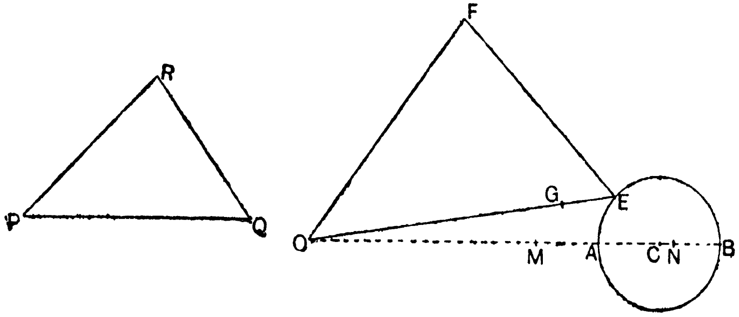
6. The points A, B, C, D, lie on the circumference of a circle: prove that the orthocentres of the triangles BCD, CDA, DAB, ABC, lie on an equal circle.

7. Prove that the pedal lines of three points on a circle with regard to any inscribed triangle form a triangle similar to that formed by the three points.

8. From a given point straight lines are drawn to the circumference of a given circle and turned through equal angles in the same direction: prove that their extremities in the new positions lie on an equal circle.

*9. A triangle similar to a given triangle has one vertex fixed, and a second moves on a given circle: prove that the third vertex also describes a circle.

Solution. Let the varying Δ be similar to the Δ PQR; let O be the fixed vertex, C the centre of the given $\odot$. Draw the diameter OACB; take OM : OA = ON : OB = PR : PQ; then the circle on MN as diameter is the locus of points which divide lines from O to the given $\odot$ in the ratio PR = PQ (Exer. 286). Now, take any pt. E on the $\odot$, draw the Δ OEF similar to the Δ PQR, and make OG = OF,



∵ $OG : OE = OF : OE = PR : PQ = OM : OA$,

∴ G lies on the ⊙ on MN as diameter.

Also $\angle GOF = \angle QPR$ and is constant, and $OF = OG$;

∴ by the preceding exercise, F moves on a ⊙ equal to the ⊙ mentioned above.

10. Equilateral triangles are drawn on the sides of a square, either all internally or all externally: compare the area of the quadrilateral formed by joining their vertices with the area of the square.

11. Draw a straight line parallel to one side of a triangle and meeting the base, so that the length of the line within the triangle is the mean proportional between the segments of the base.

12. Given the sum of the squares of two straight lines and their ratio, construct the lines geometrically.

13. Two circles, (A) and (B), intersect at M and N; a third circle meets (A) in P and Q and (B) in R and S. Prove that $MP.MQ : NP.NQ = MR.MS : NR.NS$.

14. Any points A', B', C', are taken respectively on the sides BC, CA, AB, of a triangle, and circles on AA', BB', CC' as diameters are drawn: prove that their radical centre is the orthocentre of the triangle.

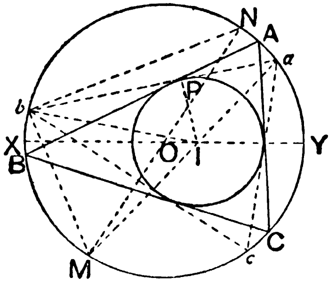
15. If D, E, F are the feet of the perpendiculars from the vertices on the sides BC, CA, AB, of a triangle, prove that

$$\Delta ABC = \tfrac{1}{2}R\,(EF + FD + DE).$$

16. The perpendiculars from the midpoints of the sides BC, CA, AB, of a triangle drawn respectively to the tangent to the circumcircle at A, AB, AC, are concurrent.

17. The incircle of a triangle touches the sides BC, CA, AB respectively at X, Y, Z: prove that the perpendiculars from the midpoints of YZ, ZX, XY respectively to BC, CA AB meet at the nine-point centre of XYZ

*18. If one triangle can be drawn so as to be inscribed in a given circle and circumscribed to another given circle, prove that any number of such triangles can be drawn.



Solution. Let I and O be the centres of two ⊙s which are such that one △ ABC can be circumscribed to the first and inscribed in the second. Take any pt. *a* on the circumference of the outer○, and draw *ab*, *ac* tangents to the inner circle: then *bc* shall also touch this ⊙.

Let *a* I meet the outer ○ in M; draw the diameter MON, join M*b* and N*b*; also join I to P, the pt. of contact of *ab* with the inner ○.

If r, R are the radii of the inner and outer ○s, we have $OI^2 = R^2 - 2Rr$ [Mis. Ex. IV, 24].

Now $MI.Ia = XI.IY = OX^2 - OI^2 = R^2 - OI^2 = 2Rr = MN.IP$.

$\therefore \frac{MI}{MN} = \frac{IP}{Ia} = \frac{Mb}{MN}$, from the similar △s I*a*P, MN*b*.

Thus $MI = Mb$; $\therefore \angle MIb = \angle MbI$;

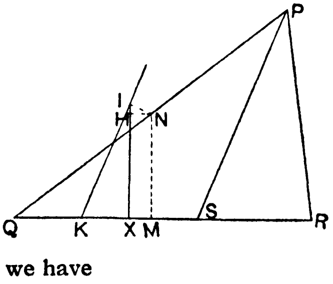
$\therefore \angle Mab + \angle Iba = \angle Mbc + \angle Ibc = \angle Mac + \angle Ibc$.

But $\angle Mac = Mab$, as *ab*, *ac* are tangents ; $\therefore \angle Ibc = \angle Iba$.

But *ab* touches the inner○ ; ∴ *bc* also touches it. [See 42.]

Hence *abc* is a similarly described triangle ; and as *a* is *any* pt. on the outer circumference, any number of such △s can be drawn.

19. PQR is a triangle right-angled at R ; a straight line parallel to the bisector of the angle P meets QR in K, and QX is taken the mean proportional to QK and QR. If XI perpendicular to QR meets the parallel at I, and N is the midpoint of PQ, prove that $NI = QN \pm IX$.



Solution. Let PS be the bisector of the ∠ P meeting QR ; M, N the midpoints of QR, QP, so that MN is perp. to QR ; draw NH parl. to QR to meet IX.

$\because \frac{QK}{QX} = \frac{QX}{QR}$, $\therefore \frac{QX}{QR} = \frac{QX - QK}{QR - QX} = \frac{KX}{XR}$;

$\therefore QR.KX = QX.XR = QM^2 - MX^2 = QM^2 - HN^2$.

Hence $HN^2 = QM^2 - 2\ QM.KX \ldots (a)$.

Now, from the similar △s IKX, PSR, we have

$$\frac{IX}{XK} = \frac{PR}{RS} = \frac{PQ}{QS} \text{ [XXII]}$$

$$= \frac{PR + PQ}{QR} \text{ [vii, § 12]} = \frac{NM + NQ}{QM} = \frac{QM(NQ + NM)}{NQ^2 - NM^2} = \frac{QM}{NQ - NM}.$$

$\therefore IX.NQ - IX.HX = QM.KX$.

Substituting in (*a*), we get

$$\begin{aligned}HN^2 &= QM^2 - 2\,IX.NQ + 2\,IX.HX\\ &= QN^2 - HX^2 - 2\,IX.NQ + 2\,IX.HX\\ &= (QN - IX)^2 - (IX - HX)^2 = (QN - IX)^2 - IH^2.\end{aligned}$$

$$\therefore\ (QN - IX)^2 = HN^2 + IH^2 = IN^2\,;\ \therefore\ IN = QN - IX.$$

If X is in RQ produced or K in QR produced, it will be found that $IN = QN + IX$. It follows that the ⊙ about the △PQR and the ⊙ whose centre is I and radius IX, have internal contact-see [43]; and the important theorem in the next exercise can be readily deduced.

*20. The nine-point circle of any triangle touches the inscribed circle and each of the escribed circles.

21. The incircle of a triangle touches the sides BC, CA, AB respectively in X, Y, Z, and XX′, YY′, ZZ′ are its diameters through the points of contact: prove that AX′, BY′, CZ′ are concurrent.

22. P is any point within a triangle ABC; straight lines AX, BY, CZ are drawn so as to make ∠XAC = ∠PAB, ∠YBA = ∠PBC, ∠ZCB = ∠PCA. Prove that AX, BY, CZ are concurrent at a point Q.
(The points P and Q are said to be *isogonally conjugate* with regard to the △ABC.)

23. In the exercise above if perpendiculars are drawn from P and Q to the sides, prove that their six feet lie on a circle.

*24. The tangents at B and C, C and A, A and B, to the circumcircle of a triangle ABC, meet in P, Q, R respectively: prove that (i) AP, BQ, CR concur at a point K; (ii) AP bisects straight lines parallel to QR and terminated by AB and AC; and (iii) if G is the centroid of the triangle, ∠GAB = ∠KAC, ∠GBC = ∠KBA.
(The point K is thus isogonally conjugate to G.)

25. O is any point in the plane of a triangle ABC; the bisectors of the angles BOC, COA, AOB, meet BC, CA, AB respectively in P, Q, R; and OP′, OQ′, OR′ perpendicular respectively to OP, OQ, OR, meet the same sides in P′, Q′, R′. Prove that (i) AP, BQ, CR are concurrent; (ii) P′, Q′, R′, as also P, Q, R′, are collinear.

26. Any circle drawn through the limiting points of a co-axal system cuts orthogonally every circle of the system.

27. The six limiting points of any three circles are concyclic.

28. If L, L′ are the limiting points of a co-axal system of circles, and C the centre of any one of them, prove that $CL.\,CL' = (\text{radius})^2$.

29. The centres of similitude of each pair of three circles form six points lying three by three on four straight lines.

30. Four points A, B, C, D are collinear in order: find the locus of points at which the segments AB and CD subtend equal angles.

31. The range (AB, PQ) is harmonic; C is the midpoint of AB and R of PQ. If V is any other point in the line, prove that $VA.VB + VP.VQ = 2VC.VR$.

32. The straight lines joining a point O to the vertices of a triangle meet the sides BC, CA, AB in P, Q, R respectively; and straight lines AP′, BQ′, CR′ are drawn to the opposite sides so as to make the pencils A (BC, PP′), B (CA, QQ′), C (AB, RR′) harmonic. Prove that P′, Q′, R′ are collinear.

33. The points A, B, C lie on a straight line; P is the harmonic conjugate of A with regard to B and C, Q of B with regard to C and A, R of C with regard to A and B. Prove that (AP, QR), (BQ, RP), (CR, PQ) are harmonic ranges.

34. The straight lines joining the extremities of a chord of a circle to the midpoint of any chord passing through its pole, are equally inclined to the latter chord.

*35. Any two points conjugate with regard to a circle and the extremities of the diameter on which they lie form a harmonic range; and any two conjugate lines and the tangents to the circle from their point of intersection form a harmonic pencil.

36. Prove that the orthocentre of a triangle is the centre of the circle with regard to which the triangle is self-conjugate. When can a real triangle be self-conjugate with regard to a circle? And what is the length of the radius of the circle?

37. Prove that the polar circle of a triangle cuts the sides of the triangle harmonically.

*38. If a quadrilateral is inscribed in a circle, the triangle formed by its three diagonals is self-conjugate with regard to the circle.

39. The circle whose diameter is the third diagonal of a quadrilateral inscribed in a circle, cuts this circle orthogonally.

40. PAP′, QAQ′ are two chords of a circle, and B is the inverse of A with regard to the circle: prove that PQ, P′Q′ subtend equal angles at B.

41. Two chords of a circle drawn through a point are at right angles, and tangents are drawn at their extremities to form a quadrilateral: prove that (i) this quadrilateral is cyclic; (ii) its diagonals pass through the given point; and (iii) the angles between the diagonals are bisected by the chords.

42. AL, BM, CN are the perpendiculars of a triangle ABC, and MN, NL, LM, respectively meet BC, CA, AB, in P ,Q, R: prove that the circumcentres of the triangles ALP, BMQ, CNR are collinear.

43. With the notation of the exercise above prove that (i) P, Q, R are collinear; (ii) the sides of the triangle LMN are respectively perpendicular to the lines joining A, B, C to the circumcentre; and (iii) the perpendiculars from A, B, C, respectively to MN, NL, LM, are concurrent.

*44. A hexagon is inscribed in a circle: prove that its three pairs of opposite sides intersect in collinear points.

45. Three alternate sides of a cyclic hexagon form a triangle, and the remaining three form another: prove that the two triangles are in perspective.

*46. A hexagon is described about a circle: prove that its three pairs of opposite vertices lie on concurrent straight lines.

TRIGONOMETRY.

CHAPTER IX.

MEASUREMENT OF ANGLES.

§ **24.** The word *Trigonometry* is derived from Greek, and signifies the measurement of triangles. The present science of Trigonometry is concerned with the general theory of angles, both geometrical and algebraical; the measurement of triangles, however, forms its most important application in practice.

An angle is measured, like any other physical quantity, by reference to an invariable unit of its own kind. One such unit is the *right angle*, which is known to be always of fixed magnitude. As however it is inconveniently large, its ninetieth part, called a *degree*, is usually employed as the unit; and the *degree measure* of any angle is obtained by finding the *ratio* of the angle to an angle of one degree. For estimating small angles the sixtieth part of a degree, called a *minute*, is employed; and this again is subdivided into sixty parts, each of which is called a *second*.

On account of the occurrence of *sixty* in the subdivisions of a degree, the method above is called the *sexagesimal* method of measuring angles. Another way of subdividing the right angle will be found in Exer. 15 below.

A different method is employed in the more algebraical treatment of the subject, called the *radian* method or the *circular measurement* of angles. The unit employed in it is the angle subtended at the centre of any circle by an arc equal to its radius; this angle is found to be invariable and is called a *radian*.

Examples. 1. Express ·6155 of a right angle in degrees, minutes and seconds.

$$\text{The given angle} = (\cdot6155 \times 90)^\circ = 55\cdot395^\circ$$
$$= 55^\circ\ 23\cdot7' = 55^\circ\ 23'\ 42''.$$

2. Find the degree measure of each angle of a regular polygon of 12 sides.

Let x be the measure required; then

$$12\,x = \text{sum of all the interior angles}$$
$$= (2 \times 12 - 4)\text{ rt. angles} = (20 \times 90)^\circ;$$
$$\therefore x = 150^\circ.$$

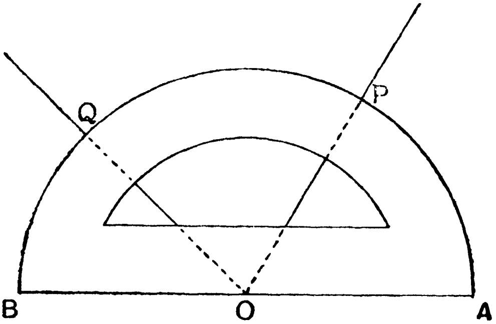
3. Reduce an angle of $50^\circ\ 37'\ 30''$ to the fraction of two right angles.

The given angle $= 50^\circ\ 37\tfrac{1}{2}' = (50\tfrac{75}{120})^\circ$

$= (\tfrac{405}{8})^\circ = \tfrac{405}{1440}$ of 2 rt. angles;

thus the result is $\tfrac{9}{32}$.

Angles are measured as well as laid off by means of the *Protractor*. This is a thin semicircular disc of metal or stout cardboard, graduated along its outer circular edge in degrees from 0 to 180. Since equal angles at the centre stand on equal arcs, the marks on the semicircle are equidistant. The diameter of the semicircle, which evidently joins the graduations for 0° and 180°, may be called the *initial line*, and its centre the *origin*, of the instrument.



In order to measure a given angle, place the origin, O, at the vertex of the angle and the initial line, AB, along one of the arms of the angle; the number at the point where the other arm of the angle meets the edge of the protractor, gives the measure of the angle (AOP or AOQ of the figure). If P does not coincide exactly with a graduation, the angle may be read to a fraction of a degree by estimation: but this is a matter of difficulty with small instruments, which should be read off to the nearest degree or half-degree at most.

To lay off an angle of given magnitude at a point of a given straight line, place O, the origin, at the point, and the part OA of the initial line along the straight line, so that the arc of the protractor is above it; prick off at the required number of degrees on the arc, and join the point thus obtained to the given point.

When two or more angles are to be compared, it is found convenient to bring one arm of each in succession along **OA**, the vertex being at **O**; then the points **P**, **Q**, ..., where the other arm of each meets the protractor's edge, at once give the measures of the angles. We should thus have angles like **AOP**, **AOQ**, **AOB** (= 180°), estimated. If the protractor is circular and transparent about the initial line, angles larger than two or even three right angles could also be laid off and estimated. Care should be taken always to measure the angles from the initial line **OA** in the direction of the graduations from **A**, going from right to left.

Examples (*Contd.*). 4. A bicycle wheel has thirty-two spokes numbered consecutively: what angle is described by a radius in passing from the 11th to the 30th spoke?

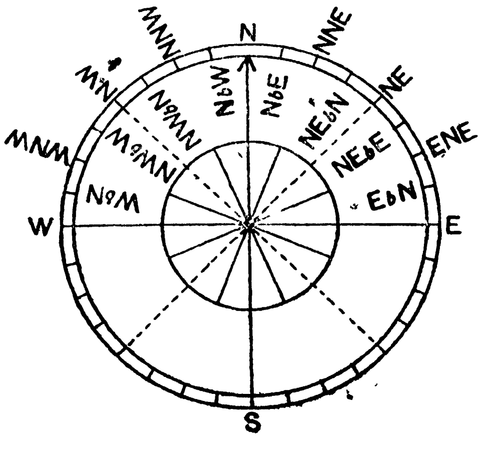
The number of angular intervals passed over is 19; also each interval is $\frac{1}{32}$ of 360° (*i.e.* the angle described by a radius in one complete revolution).

Hence the required angle $=\frac{19}{32}$ of $360^\circ = 213^\circ\ 45'$.

5. What angle is described by the minute hand of a watch between 4 o'clock and the time when the hour and minute hands are next together?

At 4 o'clock the hour hand is 120° in advance of the minute hand. If $x =$ the angle required in degrees, the hour hand describes $\frac{1}{12}x$: hence $x-\frac{1}{12}x=120^\circ$. $\therefore x=\frac{1}{11}$ of $1440^\circ = 130^\circ\ 54\frac{6}{11}'$.

As in some problems angles are indicated by reference to the points of the Mariner's Compass, the sketch of a *compass card* given in the annexed figure may prove useful to the student.



The whole circumference is divided into 32 equal parts, so that the right angle EON is divided into eight. The angle corresponding to one part is called a *point*; thus when the course of a ship is changed through $\frac{1}{8}$ of a right angle or 11° 15′, it is said to be altered by one point. The directions ENE, NNE are midway between E and NE, N and NE respectively; and N *by* E indicates the direction North altered *by one point* towards the East.

When the course of a ship or the bearing of an object from the point of observation is not exactly towards a point of the compass, fractions of a point may be employed. Thus NW by $\frac{3}{4}$ W means the direction North-West altered *by three quarters of a point* towards the West; it is therefore $53\frac{7}{16}^\circ$ West of North. This direction may also be indicated by $53\frac{7}{16}^\circ$ W of N, or by N 53° 26′ 15″ W. So also 12° 4′ North of East may be expressed as E 12° 4′ N, and North by East as N 11° 15′ E.

Exercises.

1. Write down the complementary angles of 2° 33′ and 52° 33′ 21″; and the supplementary angles of 100° 11′ and 52° 0′ 12″.

2. Find in degrees and minutes the difference in bearing between E by S and NNE, and between SW by W and NNW.

3. Simplify $7 \times 1^\circ\, 25'\, 12''$ and $\dfrac{45^\circ - 8^\circ\, 4'\, 48''}{1^\circ\, 25'\, 12''}$.

4. A triangle HLK is drawn having L bearing ENE from H, K from L 12° W of S, and H from K W by N: find the angles of the triangle.

5. One angle of a quadrilateral is half a right angle, and a second is 74° 0′ 45″; find the remaining two angles when one of them is $\frac{2}{3}$ of the other.

6. The numerical measures of the angles of a quadrilateral when referred to units containing 1°, 2°, 3°, 4° respectively are in Arithmetical Progression, and the second and fourth angles differ by a right angle: find the angles.

7. The angles of an octagon are in A. P. and the least angle contains 95°: find the remaining angles.

8. Find in minutes of angle the angle between the hands of a clock at half past five, quarter to nine and ten minutes past eleven.

9. Find the angle described by the hour hand of a watch from 3 a. m. to 1-50 p. m.

*10. The circumference of the Earth is 24900 miles, and two places 10 miles apart lie on the same meridian: find the difference of their latitudes.

11. Find the distance in miles between two places on the equator of the Earth which differ in longitude by $6^\circ\ 18'$.

12. An arctic explorer reaches a latitude of $88^\circ\ 0'\ 30''$: find how many miles he is distant from the pole, supposing the Earth to be a sphere of diameter 7980 miles.

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13. At two consecutive milestones on a straight road running NNE the bearings of a distant object are $8^\circ\ 20'$ N of E and $16^\circ\ 40'$ S of E respectively : find the distance of the object from the first milestone, by drawing the figure to a convenient scale and measuring with the ruler.

14. A ship is due W of a lighthouse and sails on a straight NNW course ; when it has proceeded 12 miles the lighthouse is found to bear SE 10° N. Find by means of a diagram the distance of the lighthouse from the second point of observation.

*15. In the *centesimal* measurement of angles the following table is employed :—

1 rt. angle = 100 grades (g), 1 grade = 100 minutes ($`$),
1 minute = 100 seconds ($``$).

Find the centesimal measures of $\frac{11}{64}$ of a rt. angle, of 100° and $10^\circ\ 10'\ 10''$.

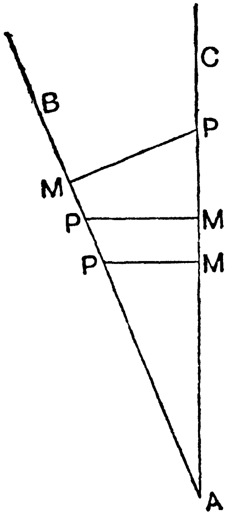
16. Reduce $50^g\ 2`\ 50``$ to the decimal of two right angles, and $42^g\ 2`\ 25``$ to sexagesimal measure.

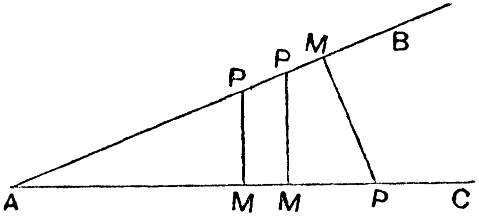
17. The number of degrees in one angle of a regular polygon is equal to the number of grades in one angle of a regular polygon of half the number of sides: find the number of the sides of each polygon.

CHAPTER X.

TRIGONOMETRICAL RATIOS.

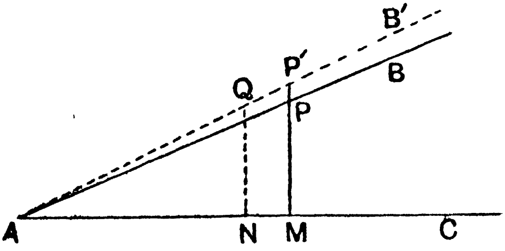
§ 25. Acute Angles.





Take an acute *angle* BAC of fixed magnitude in any position, and from *any* point P in *one* of its bounding arms draw PM perpendicular to the *other* arm. It is seen that the triangle APM thus formed has always the *same angles*, and therefore that its corresponding sides have the *same ratios* when taken two and two [Prop. XXVI]. We have thus six fractions, like $\frac{MP}{AP}$ and $\frac{AP}{MP}$, constant in value for *all positions* of P, referring to the angle BAC.

If the angle changes ever so little, each of these ratios will change. Let the angle increase to B′AC; draw P′PM perp. to AC, on AB take AQ = AP, and draw QN perp. to AC.



It is evident that Q lies between A and P′ at a greater distance from AC than P; hence

$$\frac{MP}{AP} < \frac{NQ}{AP}, < \frac{NQ}{AQ}, \; i.e. < \frac{MP'}{AP'}. \qquad \text{[XXVI]}$$

Thus these corresponding ratios of the two angles are different; so also for any other pair of corresponding ratios.

Conversely, if the corresponding ratios for two angles are equal, the angles themselves are equal, as is seen from Prop. XXVIII and Exer. 270 (§ 15). So also, if the ratios differ the angles are different.

The ratios being thus intimately connected with the angle, it is found that the properties of the angle are readily investigated by means of them. The following *fundamental* definitions are, consequently, adopted in Trigonometry.

Defs. If BAC is any given acute angle, P any point in either arm and PM perpendicular to the other arm, then the ratio

$\frac{MP}{AP}$ is called the **sine**, $\frac{AM}{AP}$ the **cosine**,

$\frac{MP}{AM}$ the **tangent**, $\frac{AM}{MP}$ the **cotangent**,

$\frac{AP}{AM}$ the **secant**, and $\frac{AP}{MP}$ the **cosecant**,

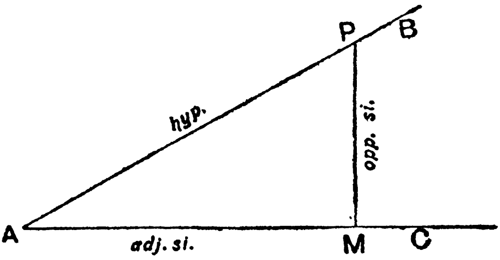
of the angle BAC.

As these ratios are continually employed in all branches of Mathematics, it is found very convenient to use the following respective abbreviations for them, denoting by A the value of the angle :—

*sin*A, *cos*A, *tan*A, *cot*A, *sec*A, *cosec*A.

The symbol sin A means "the sine of the angle A", and is not divisible into parts; so for the other symbols. These ratios are said to be the *trigonometrical ratios* or *functions* of the angle.

If we take the triangle APM formed by the perpendicular, we see that AP is the hypotenuse, and AM and MP the sides respectively adjacent and opposite to the given angle BAC. Hence, *in the present case*, we can also define thus:— construct the triangle APM, right-angled at M, to contain the given angle A; then



$$\sin A = \frac{\text{opp. side}}{\text{hypot.}},\quad \cos A = \frac{\text{adj. side}}{\text{hypot.}},\quad \tan A = \frac{\text{opp. side}}{\text{adj. side}},$$

$$\cot A = \frac{\text{adj. side}}{\text{opp. side}},\quad \sec A = \frac{\text{hypot.}}{\text{adj. side}},\quad \operatorname{cosec} A = \frac{\text{hypot.}}{\text{opp. side}}.$$

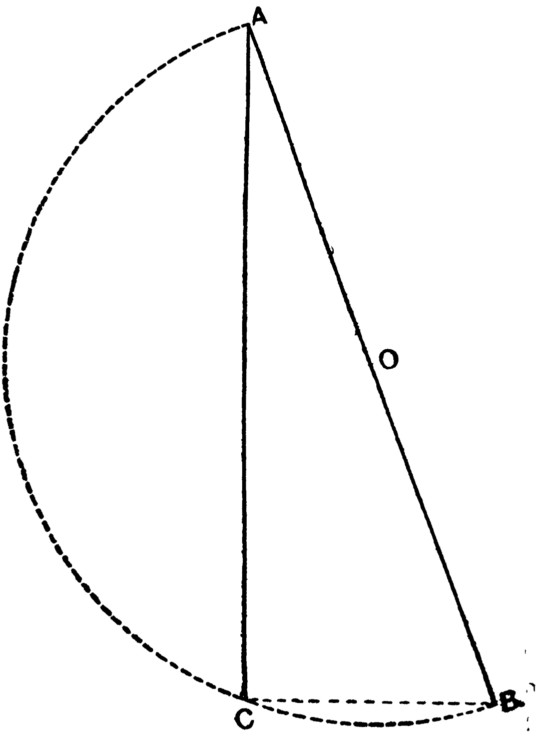
Examples. 1. To find the sine of an angle of 37°.

Take a straight line AC, and at A lay off the angle (37°). In the other arm take P so that AP = 100 mm., and draw PM perp. to AC; it is found on measurement that MP = $60\frac{1}{2}$mm. Hence $\sin 37^\circ = \frac{60\frac{1}{2}}{100} = \cdot 605$.

By reading off the length AM, we can similarly find cos 37°, tan 37°, &c.

The larger the figure is drawn the greater will be the accuracy of the numbers found. But the construction will give the ratios at most to two places of decimals, as even with careful drawing we cannot lay off angles to within half a degree or lengths to $\frac{1}{100}$ of an inch. The values of the trigonometrical ratios are more readily and quite accurately calculated by means of formulæ given in the higher parts of the subject, and entered in Tables like those given at the end, whence we obtain sin 37° = ·6018.

2. To construct the angle whose sine is ·34.



Take a ○ of diameter AOB = 100 mm.; from B draw the chord BC of length 34 mm. Then BAC is the angle required.

As ∠ ACB = 90° in a semi-⊙,

$\therefore \quad \sin A = \frac{CB}{AB} = \frac{34}{100} = \cdot 34.$

On measuring with the protractor, the angle is found to be almost exactly 20°. The Tables give 19°53′.

An angle with a given cosine or cosecant may be similarly constructed. The construction when the tangent is given is different, as may be seen from the following example.

3. The sides about the right angle of a right-angled triangle are 31 and 22 : find its acute angles.

Let the $\triangle APM$ (see fig., p. 159) have a right angle at M, and $AM = 31$, $MP = 22$. Measuring with the protractor, we get $\angle A = 35^\circ$ and $\angle P = 65^\circ$.

Or, reducing $\frac{22}{31}$ to a decimal, we have $\tan A = \cdot 7097$; hence $\angle A = 35^\circ\ 22'$ from the Tables, and $\therefore \angle P = 64^\circ\ 38'$.

It is seen that such examples could be solved without the use of trigonometrical tables ; but the results would not be very accurate as pointed out above.

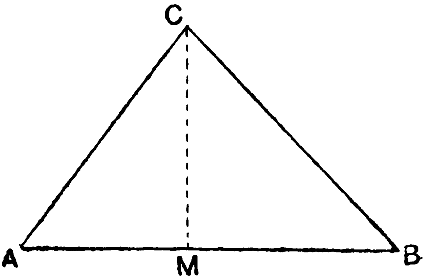
4. A kite is flying at the end of a string 100 ft. long : if the string is taut and makes an angle of 40° with level ground, find the height of the kite.

Taking the usual figure, we have now $AP = 100$, $\angle A = 40^\circ$; and MP is the height required.

$$\text{Now } \frac{MP}{AP} = \sin A; \therefore MP = AP \sin A = 100 \sin 40^\circ$$
$$= 100 \times \cdot 6428 \text{ [Tables]}$$
$$= 64\tfrac{1}{4} \text{ (very nearly).}$$

5. Two sides of a triangle are $1\frac{1}{2}$ and 2 in., and the included angle is 53° : find its area.

In the figure, let $AB = 2$, $AC = 1 \cdot 5$, $\angle A = 53^\circ$. Draw CM perp. to AB, and let Δ denote the area of the triangle; then $\Delta = \frac{1}{2}$ rect. AB.MC.



$$\text{But } \frac{MC}{AC} = \sin 53^\circ; \therefore MC = AC \sin 53^\circ,$$
$$\text{and } \Delta = \tfrac{1}{2} AB.AC \sin 53^\circ \text{........(1).}$$

Hence, from the Tables, $\Delta = \frac{1}{2}(3)(\cdot 7986) = 1 \cdot 1979$; so that the area is very nearly $1 \cdot 2$ sq. in.

It is very useful to remember equation (1), which shows that *the area of a triangle*

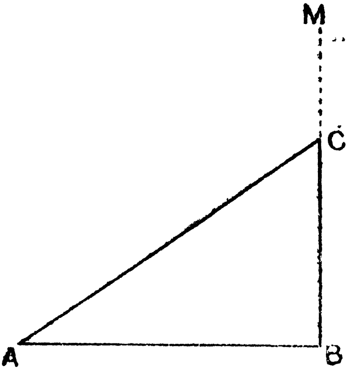
$= \frac{1}{2}$ (*rect. of two sides*) (*sine of included angle*).

6. The hypotenuse of a right angled triangle is 2 ft. and the cosine of one of its acute angles is $\frac{5}{6}$: find the angles and the sides of the triangle.

As cosine $= \dfrac{\text{adj. side}}{\text{hypot.}}$, and as here hypot. $= 2$, the adj. side is $\frac{5}{3}$.

Take AB to represent $1\frac{2}{3}$; draw BM at rt. angles to AB. With radius 2 and centre A describe a ○, cutting BM in C. Then ABC is evidently the right-angled triangle given.

On measuring, we find $\angle A = 34^\circ$, $\angle C = 56^\circ$; also $BC = 1{\cdot}1$.



The second side may be thus found :— $\because \dfrac{BC}{AC} = \sin 34^\circ$,

$\therefore \quad BC = AC \sin 34^\circ = 2({\cdot}559) = 1{\cdot}12$ nearly, from the Tables.

Or again thus :— $BC^2 = AC^2 - AB^2 = 4 - \frac{25}{9} = \frac{11}{9}$;

$\therefore \quad BC = \frac{1}{3}\sqrt{11} = 1{\cdot}106$ nearly.

Exercises.

18. Draw an angle of 72° and measure the value of its sine. Compare with the result given in the Tables.

19. Find as above $\cos 24^\circ 30'$, $\tan 56^\circ$, $\sec 74^\circ$.

20. AD is drawn perpendicular to BC, the hypotenuse of a right-angled triangle ABC : find three expressions for sin B , and prove them equal. Also write down tan C in three different ways.

21. Find from the Tables the angles with sine $\frac{3}{5}$, cosine $\frac{3}{5}$, tangent 2, cotangent $\frac{1}{2}$. What relation is there between the first and second angles, and between the third and fourth?

22. Construct angles A and B such that $\sin A = {\cdot}73$, $\tan B = 1{\cdot}4$; and measure them to the nearest half degree.

23. Find from the Tables $\sec 40^\circ$, $\cot 30^\circ 30'$, $\operatorname{cosec} 50^\circ 55'$.

24. Construct in two ways the angle whose sine is $\dfrac{a}{b}$.

25. Construct the angle whose secant is $\frac{29}{20}$, and from the figure read off its other trigonometrical ratios.

26. An isosceles triangle on a base 24″ long has equal sides 20″ long: find the value of either base angle, and write down its sine and cotangent.

*27. The diagonals of a quadrilateral are of lengths e, f, and include an angle α: prove that its area is $ef \sin\alpha$.

28. A rhombus has an angle of 55° and each side of 2″; find its area and the lengths of its diagonals, having given $\sin 55^\circ = \cdot 819$, $\sin 27^\circ 30' = \cdot 462$ and $\cos 27^\circ 30' = \cdot 887$.

29. A ladder leaning against a wall reaches to a point 16 ft. vertically above the level ground, with which it makes an angle of 73°: find the length of the ladder.

30. A column 32 ft. high casts a shadow of length 26 ft: find the altitude of the sun.

31. A railway train is moving up a slope of $1^\circ 30'$: find the height through which it ascends in going $1\frac{1}{2}$ miles.

32. A flagstaff subtends an angle of 20° at a point distant 20 metres from its foot: find the height of the flagstaff.

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33. From a point P, distant 2·5 cm. from the centre (O) of a circle of radius 1·75 cm, two tangents PS, PT are drawn to the circle. Find the trigonometric tangent of the angle TPO, and thence find the angle between the tangents by means of the Tables.

34. On a straight line 1·5″ long a segment of a circle is described to contain an angle of 51°: calculate the radius of the circle.

35. From the fact that each angle of a regular pentagon contains 108°, obtain the radius of the circumcircle of a regular pentagon of side 2 cm.

*36. Prove that the area of a regular hexagon inscribed in a circle of radius 18 mm. may be expressed as

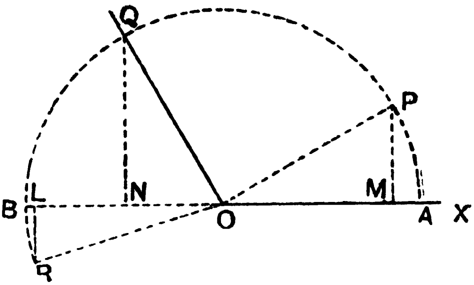
$$972 \sin 60^\circ \text{ or } 486 \cot 30^\circ \text{ (sq. mm.)}.$$

37. The circumference of a circular ring, supported horizontally and having a diameter of 4 ft, is divided into eight equal parts; from a point 5 ft. vertically above the centre of the ring, straight strings are attached to the points of division. Find the inclination (i) of each string to the vertical through the point, (ii) of two consecutive strings to each other.

§ 26. OBTUSE ANGLES : RATIOS OF RELATED ANGLES.

We now proceed to consider the trigonometrical ratios of obtuse angles. Taking any such angle XOQ, place its vertex at O the centre of the protractor, and one of its arms along OA the diameter: then the other arm of the angle will cut the arc of the protractor in some point, Q, and AOQ will give the measure of the angle. Draw QN perpendicular to AOB : then, as in the preceding section, the six ratios

$\frac{NQ}{OQ}$, $\frac{ON}{OQ}$, $\frac{NQ}{ON}$, $\frac{ON}{NQ}$, $\frac{OQ}{ON}$, $\frac{OQ}{NQ}$, are respectively called the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, of the angle XOQ.



It is seen that these ratios remain fixed if the angle is given, no matter how the angle is drawn or where the point Q is taken in the radius OQ. The point might be taken even in the arm OX and a perpendicular from it drawn to QO produced. If the obtuse angle changes, each of the trigonometrical ratios will change.

When the angle is acute, as AOP, the foot of the perpendicular PM lies *in* OX *towards* A ; but for the obtuse angle AOQ, the foot of the perpendicular PN lies *on* XO *produced towards* B. This is the fundamental *difference* in the figures for acute and obtuse values of the angle. Both perpendiculars, NQ and MP, are seen to be drawn on the *same side* of AOB, here above it.

After bringing the angle, AOP or AOQ, into position, and drawing the perpendiculars PM, QN, let these perpendiculars as well as OM and ON be measured as in the graphical representation of points. Then we see that MP, QN are *positive* for both *acute* and *obtuse* angles, but that OM is *positive* for *acute* angles while ON is *negative* for *obtuse* angles. (See the explanations and illustration 1 in § 4.)

The acute angle XOP is an angle of the triangle OPM ; but the obtuse angle XOQ is not, and in fact cannot possibly be, an angle of the triangle OQN. We *may not* therefore define its trigonometrical ratios in terms of *base* and *opposite side* as in the previous section.

The perpendiculars MP, NQ correspond to the *ordinates* of the *points* P, Q, as defined in graphical constructions, and the intercepts of the diameter, OM, ON, to the *abscissæ* of these points. Hence we may give the following general definitions of the ratios of any angle. Bring the angle into position as AOP; take any point P on the arm OP, and find its co-ordinates, and call OP the *revolving length* : then

$$\sin(\angle)=\frac{ord.}{r.\ l.},\quad \cos(\angle)=\frac{abs.}{r.\ l.},\quad \tan(\angle)=\frac{ord.}{abs.},$$

$$\cot(\angle)=\frac{abs.}{ord.},\quad \sec(\angle)=\frac{r.\ l.}{abs.},\quad \operatorname{cosec}(\angle)=\frac{r.\ l.}{ord.}.$$

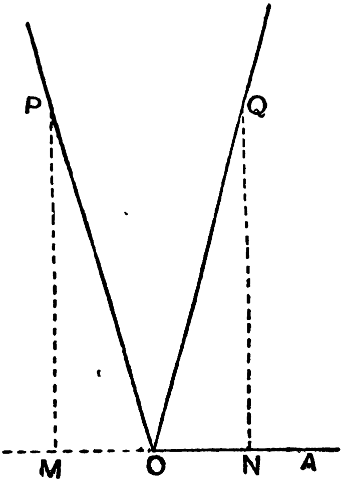
It will be seen that we could even take the angle to be greater than two right angles, by moving OQ further so as first to coincide with OB and then pass beyond it, the total angle formed being the reflex angle XOR ; the functions of the angle may then be defined in terms of LR, OL, the co-ordinates of R, and the revolving line OR.

Examples. 1. Find the ratios for an angle of 105.°

Let OA represent the base-radius of the protractor; make ∠ AOP = 105,° and take OP = 50 mm.; draw PM perp. to AO produced. On measuring carefully, PM is found to be 48·5 and OM to be 13.

Hence $\sin 105^\circ = \frac{48.5}{50} = \cdot 970,$ as MP is + ve ; and $\cos 105^\circ = -\frac{13}{50} = -\cdot 260$, as OM is − ve.

We can similarly find $\tan 105^\circ\ (= -3\cdot 731)$ and the other ratios.



2. To compare the ratios of an angle of 75° with those of 105.°

In the figure above make ∠ AOQ = 75°, take OQ = 50 mm., and draw QN perp. to OA.

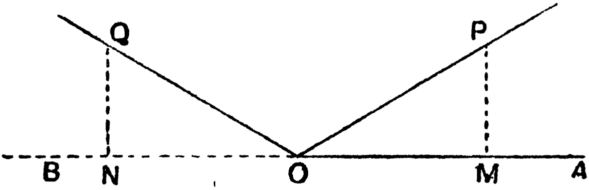
As ∠ AOP = 105°, ∴ ∠ MOP = 180° − 105° = 75° = NOQ ;
hence the Δ s NOQ, MOP are congruent.

Thus $\sin 75^\circ = \frac{NQ}{OQ} = \frac{MP}{OP} = \cdot 970$;

and $\cos 75^\circ = \frac{ON}{OQ} = \frac{-OM}{OQ} = +\cdot 260$; and so on for the other ratios.

We have shown that the ratios of an acute angle change when the angle changes, so that no two acute angles have any one ratio the same for both. But if we compare an obtuse angle with an acute angle, as in the preceding example, it may happen that the triangles OPM, OQN formed for the two are congruent, and then some of the ratios may be the same for both. In this case it will be found that the angles bear certain relations to each other; and we are thus led to the following investigations.

I. *To compare the trigonometrical ratios of any angle and its supplement.*



Let AOB be the diameter and O the centre of the protractor. Mark off $\angle AOP = A^\circ$ and $\angle AOQ = (180 - A)^\circ$, the supplementary angle, so that $\angle BOQ = A^\circ$ numerically. Take $OP = OQ$, and draw the perps. PM, QN to AOB.

Then the Δs POM, QON are congruent.

Hence, the perps. MP, NQ are equal and of like signs; while the bases OM, ON, are equal but of unlike signs.

$$\therefore \mathbf{sin}\ (\mathbf{180 - A})^\circ = \sin AOQ = \frac{NQ}{OQ} = \frac{MP}{OP} = \mathbf{sin\ A^\circ};$$

and $$\mathbf{cos}\ (\mathbf{180 - A})^\circ = \frac{ON}{OQ} = \frac{-OM}{OP} = -\mathbf{cos\ A^\circ}.$$

Similarly we see that

$$\tan (180 - A)^\circ = -\tan A^\circ, \qquad \cot (180 - A)^\circ = -\cot A^\circ,$$
$$\sec (180 - A)^\circ = -\sec A^\circ, \qquad \operatorname{cosec} (180 - A)^\circ = \operatorname{cosec} A^\circ.$$

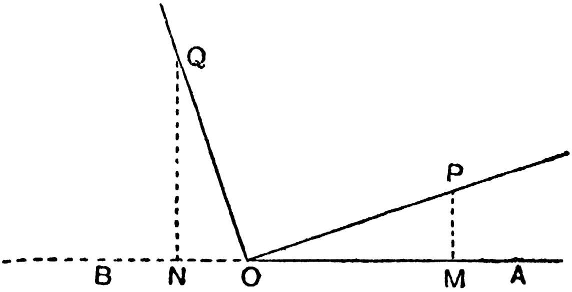
Thus the ratios of the supplement of any angle have the same numerical values as the corresponding ratios of the angle; the *sine* and *cosecant* are of the *same sign* for both, the remaining ratios have *opposite signs.*

It follows that if the cosine or tangent of an angle is a given negative quantity, we should find from the Tables the angle with the corresponding positive cosine or tangent, and take its supplement as the value of the angle.

If ∠ AOQ is denoted by A (in degrees), ∠ AOP will be 180 − A, *i. e.* the supplementary angle: hence the relations given above are *mutual,* as is seen from the formulæ. It is found that they hold good even when A is a reflex angle.

II. *To compare the trigonometrical ratios of an angle of* A° *and of* (90 + A)°

Drawing the centre-line of the protractor, mark off ∠ AOP = A°, and take ∠ POQ (in the same way round the arc) = a right angle; then ∠ AOQ = (90 + A)°.



Set off OQ = OP, and draw the perps. PM, QN.

∵ ∠ POQ = 90°, ∴ ∠ POM + ∠ QON = 90°;
but ∠ NQO + ∠ QON = 90° also, ∴ ∠ POM = ∠ NQO.

Thus the △s POM, QON are congruent;
hence NQ = OM, and they have like signs,—
while ON = − MP as they have unlike signs.

$$\therefore \sin (90+A)^\circ = \frac{NQ}{OQ} = \frac{OM}{OP} = \cos A^\circ,$$

$$\text{and } \cos (90+A)^\circ = \frac{ON}{OQ} = \frac{-MP}{OP} = -\sin A^\circ.$$

Similarly it is seen that

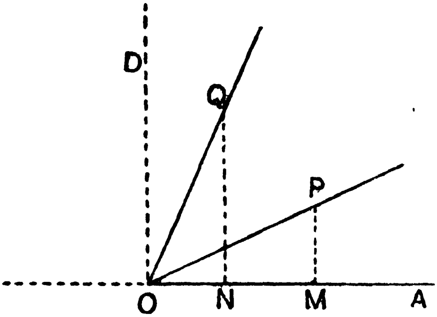
$$\tan (90+A)^\circ = -\cot A^\circ,\ \cot (90+A)^\circ = -\tan A^\circ,$$
$$\sec (90+A)^\circ = -\operatorname{cosec} A^\circ,\ \operatorname{cosec} (90+A)^\circ = \sec A^\circ.$$

Thus the trigonometrical ratios of 90 + A are found by interchanging *sin* and *cos*, *tan* and *cot*, *sec* and *cosec*, and keeping the signs of *sin* and *cosec positive* and changing the signs of the remaining ratios to *negative*.

These formulæ are found to hold good when A is obtuse and consequently 90 + A is reflex.

III. T*o compare the trigonometrical ratios of an angle and its complement.*

As before, mark off $\angle$ AOP $=$ A° (acute); draw OD perp. to OA, and make $\angle$ DOQ $=$ A° within the right angle AOD. Then $\angle$ AOQ $= \angle$ AOD $- \angle$ DOQ $= (90 - \text{A})°$. Set off OQ $=$ OP, and draw the perps. PM, QN.



$\because \angle \text{QON} = (90 - \text{A})° = 90° - \angle \text{POM} = \angle \text{OPM}$,

$\therefore$ the Δs POM, QON are congruent.

Hence NQ, OM are equal in magnitude and sign; and so also are ON and MP.

$$\therefore \ \sin\,(90 - \text{A})° = \frac{\text{NQ}}{\text{OQ}} = \frac{\text{OM}}{\text{OP}} = \cos \text{A}°,$$

$$\text{and} \ \cos\,(90 - \text{A})° = \frac{\text{ON}}{\text{OQ}} = \frac{\text{MP}}{\text{OP}} = \sin \text{A}°.$$

Similarly we get

$\tan\,(90 - \text{A})° = \cot \text{A}°$, $\cot\,(90 - \text{A})° = \tan \text{A}°$,
$\sec\,(90 - \text{A})° = \text{cosec}\, \text{A}°$, $\text{cosec}\,(90 - \text{A})° = \sec \text{A}°$.

Thus the trigonometrical ratios of the complement of any angle are found from those of the angle by interchanging the *sine* and *cosine*, the *tangent* and *cotangent*, the *secant* and *cosecant*, *without any change of sign.* If the angle AOQ is A, then AOP is $90 - \text{A}$; the relations given above are, therefore, mutual, as is seen from the formulæ also. These results explain the origin of the prefix *co* in three of the ratios: for the *cosine of* A is $\frac{\text{OM}}{\text{OP}}$, and is the same as $\frac{\text{NQ}}{\text{OQ}}$ the *sine of the complementary angle.* So also, the *co*tangent of A is the tangent of the *complement*, and the *co*secant of A the secant of the *complement.*

The formulæ may be proved from the triangle OPM itself. If $\angle$ POM = A, $\angle$ OPM = 90 − A;

$$\text{and}\quad \sin(90-A)=\sin OPM=\frac{\text{opp. side to }\angle P}{\text{hypotenuse}}$$

$$=\frac{OM}{OP}=\frac{\text{adj. side to }\angle O}{\text{hypotenuse}}=\cos A.$$

Also $\cos(90-A)=\frac{MP}{OP}=\sin A$; and so for the other ratios.

Exercises.

38. Construct an angle of 144°, and find all its trigonometrical ratios.

39. Find from the Tables $\sin 47^\circ 30'$; and hence deduce $\sin 132^\circ 30'$, $\cos 137^\circ 30'$.

40. By means of the Tables find the angle whose cosine is $-{\cdot}4848$, and that whose tangent is $-2{\cdot}1693$.

41. Given $\sin A=a$, $\cos A=b$; find the sine of the supplement of the complement of A, and the cotangent of the supplement of the supplement of A.

42. Prove that the sine of an angle is greater than the cosine when the angle lies between 45° and 135°; but that when the angle lies between 0° and 45° or 135° and 180°, the cosine is numerically greater than the sine.

*43. Trace the changes in the value of the sine of an angle which varies, by finding the sines of 5°, 10°, 15°,.. 175° from the Tables. Find also the variation of the cosine.

44. Given $\sin 15^\circ=\frac{1}{4}(\sqrt{6}-\sqrt{2})$, $\cos 15^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})$, find all the trigonometrical ratios of 105° and 165°.

45. Given $\tan 49^\circ=1{\cdot}15$, find by geometrical construction the sine and cosine of 51° and 131°. Verify the results by referring to the Tables.

*46. If A, B, C are the measures of the angles of a triangle, prove that

$$\sin(A+B)=\sin C,\qquad \cos(A+B)=-\cos C,$$

$$\cos\frac{A+B}{2}=\sin\frac{C}{2},\qquad \tan\frac{A+B}{2}=\cot\frac{C}{2}.$$

CHAPTER XI.

TRIGONOMETRICAL RATIOS OF SPECIAL ANGLES.

§ **27**. The ratios of some angles are constantly required in the subject, and the student should make himself perfectly familiar with them. This is best done by retaining in memory the figures of the right-angled triangles which contain the special angles severally.

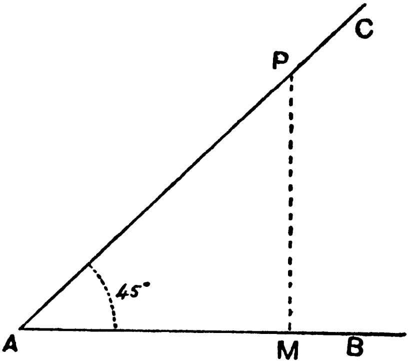
Ratios of an angle of 45° ($\frac{1}{2}$ *of a right angle*).

Let $\angle \text{BAC} = 45^\circ$. From P in AC draw PM perp. to AB; then $\angle \text{APM} = 45^\circ$.

$\therefore$ $\text{AM} = \text{MP}, = a$ suppose.

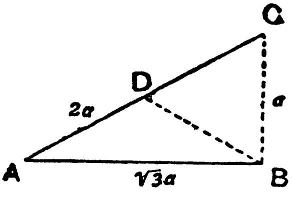
Hence $\text{AP}^2 = \text{AM}^2 + \text{MP}^2 = 2a^2$; $\therefore \text{AP} = \sqrt{2}.a$.

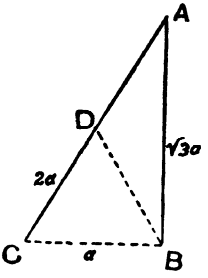
Then **sin** $45^\circ = \frac{\text{MP}}{\text{AP}} = \frac{a}{\sqrt{2}.a} = \frac{1}{\sqrt{2}}$; **cos** $45^\circ = \frac{1}{\sqrt{2}}$.



Also tan $45^\circ = 1$, cot $45^\circ = 1$, sec $45^\circ =$ cosec $45^\circ = \sqrt{2}$.

Ratios of angles of 30° *and* 60° ($\frac{1}{3}$ *and* $\frac{2}{3}$ *of a right angle*).





Let $\angle \text{BAC} = 30^\circ$. From C draw CB perp. to AB; then $\angle \text{ACB} = 60^\circ$. Draw BD to make $\angle \text{ABD} = 30^\circ$.

$\therefore$ $\angle \text{CBD} = 60^\circ$, and hence $\angle \text{BDC} = 60^\circ$ also.

Thus the Δ BCD is equilateral, and $\text{BD} = \text{DC} = \text{CB}$.

But BD = AD, as ∠ DAB = ∠ ABD ;

∴ AC = 2BC, = $2a$ suppose.

Hence $AB^2 = AC^2 - BC^2 = 4a^2 - a^2 = 3a^2$, ∴ $AB = \sqrt{3}.a$.

$$\therefore \sin 30^\circ = \frac{BC}{AC} = \frac{a}{2a} = \frac{1}{2}\ ; \cos 30^\circ = \frac{AB}{AC} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}.$$

Also $\tan 30^\circ = \frac{1}{\sqrt{3}}$, $\cot 30^\circ = \sqrt{3}$, $\sec 30^\circ = \frac{2}{\sqrt{3}}$, $\operatorname{cosec} 30^\circ = 2$.

Similarly, the second figure in which ∠ BCA = 60°, gives

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}\ ;$$

$\tan 60^\circ = \sqrt{3}$, $\cot 60^\circ = \frac{1}{\sqrt{3}}$, $\sec 60^\circ = 2$, $\operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}$.

In fact the angles of 30° and 60° are complementary, so that the sine and cosine of one are respectively the cosine and sine of the other, as proved generally in the preceding article.

Let a magnitude x diminish indefinitely as it approaches a certain point and vanish at the point ; and let a be a finite fixed magnitude of the same kind.

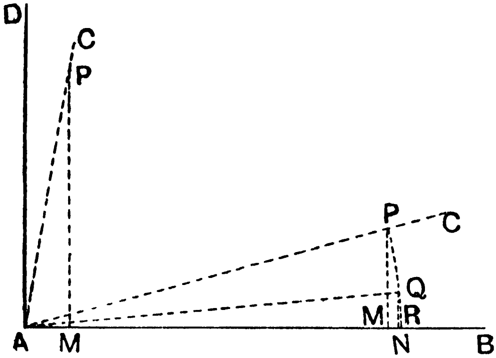
Then $\frac{x}{a}$ will be *smaller* than any assignable quantity as x comes very near the point, and will become *zero* at the point: while $\frac{a}{x}$ will then be *larger* than any assignable quantity, and will become *infinitely large* at the point. These results are thus shown—

$$\frac{x}{a} = 0 \text{ and } \frac{a}{x} = \infty, \text{ when } x = 0.$$

It should be remembered that ∞ does not stand for a *definite magnitude* ; all that can be asserted about it is that it is *larger* than any magnitude which *can be named.* Hence, if y and z both tend to become infinite, we cannot assert that $y = z$, as there is no *definite* limit to their increase. A similar remark applies to *zero* ; for if b is another definite magnitude, $\frac{x}{b}$ will also become zero when x vanishes, but $\frac{x}{b}$ will *not* bear a *ratio of equality* to $\frac{x}{a}$ unless a and b are coincident.

Illustration. If $x = \frac{1}{10^{10}}, \frac{1}{10^{13}}, \frac{1}{10^{16}}, \ldots$ as it approaches a certain point, $\frac{x}{a}$ becomes $\frac{a}{10^{10}}, \frac{a}{10^{13}}, \ldots$, and becomes zero if the process is continued long enough. On the other hand, $\frac{a}{x}$ becomes $10^{10}\,a$, $10^{13}\,a$, $10^{16}\,a, \ldots$ and becomes larger than any conceivable quantity in the limit when x vanishes.

Ratios of angles of 0° and 90.°



Take a very small angle BAC, with centre A and radius $= a$, draw the arc PQR cutting the arms in P and R. Let PM or QN, perp. to AB, be denoted by p.

$$\text{Then } \sin \text{BAC} = \frac{\text{MP}}{\text{AP}}, \quad \sin \text{BAQ} = \frac{\text{NQ}}{\text{AP}};$$

these have the value $\frac{p}{a}$, which is small when p is small, and which becomes indefinitely smaller as P moves through Q towards R. In the limit, PM diminishes towards zero, until, when P actually coincides with R, PM *i. e.* p becomes zero. But in this case AC coincides with AB and the angle is also zero. We thus see that **sin** $\mathbf{0^\circ = 0}$.

Again, when ∠ A vanishes, AM or AN becomes coincident with AR; thus **cos** $\mathbf{0^\circ} = \frac{a}{a} = \mathbf{1}$.

Also $\tan \text{A} = \frac{\text{MP}}{\text{AM}} = \frac{p}{a}$ in the limit;

$\therefore$ **tan** $\mathbf{0^\circ = 0}$.

As to cot A, it is $\frac{\text{AM}}{\text{MP}}$, *i.e.*, in the limit, $\frac{\text{AP}}{\text{MP}}$ or $\frac{a}{p}$, and therefore is infinitely large when p becomes zero, as explained above: thus **cot** $\mathbf{0^\circ = \infty}$.

So also we have sec $0^\circ = 1$, cosec $0^\circ = \infty$.

Similarly, by taking an angle very near to a right angle and increasing it to 90°, we can prove that

$$\mathbf{\sin 90^\circ = 1, \quad \cos 90^\circ = 0, \quad \tan 90^\circ = \infty,}$$

$$\mathbf{\cot 90^\circ = 0, \quad \sec 90^\circ = \infty, \quad \text{cosec}\, 90^\circ = 1.}$$

Or these results may be inferred from the fact that the angles of 90° and 0° are complementary.

The triangle APM has *hyp.* $=a$, *base* $=a$, *perp.* $=0$, when the angle is zero; and *hyp.* $=a$, *perp.* $=a$, *base* $=0$, when the angle is a right angle.

Examples. 1. Prove that

$$2(\sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ) = \sqrt{2}(\cos 30^\circ + \sin 30^\circ).$$

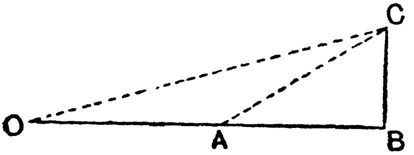
Substituting the values of the functions found above, we get

$$\text{the left side} = 2\left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right) = \frac{\sqrt{3}+1}{\sqrt{2}};$$

$$\text{the right side} = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\sqrt{3}+1}{\sqrt{2}}.$$

Thus the identity of the two sides is proved.

2. *To find the trigonometrical ratios of an angle of 15°.*



Take $\angle$ BAC $= 30^\circ$, and draw CB perp. to AB. Produce BA to O, making AO $=$ AC, and join OC. Then evidently $\angle$ BOC $= 15^\circ$.

Let BC $= a$, then, as proved before, AB $= \sqrt{3}a$, AC $= 2a$.

Hence OB $= (2 + \sqrt{3})\,a$;

and $\text{OC}^2 = \text{OB}^2 + \text{BC}^2 = a^2(7 + 4\sqrt{3} + 1) = a^2(8 + 2\sqrt{12})$.

$\therefore$ OC $= (\sqrt{6} + \sqrt{2})\,a$.

Thus the three sides of the $\triangle$ OBC are known, and the ratios of the angle at O can be written down.

$$\text{We get } \sin 15^\circ = \frac{\text{BC}}{\text{OC}} = \frac{1}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} \text{ or } \frac{\sqrt{3}-1}{2\sqrt{2}};$$

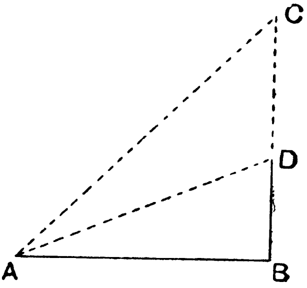
$$\cos 15^\circ = \frac{2+\sqrt{3}}{\sqrt{6}+\sqrt{2}} = \frac{8+4\sqrt{3}}{4(\sqrt{6}+\sqrt{2})} = \frac{\sqrt{6}+\sqrt{2}}{4} \text{ or } \frac{\sqrt{3}+1}{2\sqrt{2}};$$

$$\tan 15^\circ = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3},\ \cot 15^\circ = 2 + \sqrt{3}.$$

The secant and cosecant can be similarly found.

In the following example we employ a construction depending on Prop. XXII.

3. *To find the ratios for an angle of* $22^\circ 30'$ *(*$\frac{1}{4}$ *of* 90°*).*



Take $\angle$ BAC $= 45^\circ$; draw CB perp. to AB, and bisect $\angle$ A by AD meeting CB. Then $\angle$ BAD $= 22\frac{1}{2}^\circ$, and if AB $= a$, BC $= a$ and AB $= \sqrt{2}.a$.

Also by Geometry $\frac{\text{BD}}{\text{DC}} = \frac{\text{BA}}{\text{AC}}$;

$$\therefore \frac{\text{BD}}{\text{BA}} = \frac{\text{DC}}{\text{AC}} = \frac{\text{BD}+\text{DC}}{\text{BA}+\text{AC}} = \frac{\text{BC}}{\text{BA}+\text{AC}} = \frac{a}{a+\sqrt{2}a}.$$

Hence $\text{BD} = \frac{a}{1+\sqrt{2}} = (\sqrt{2}-1)a$;

and $\text{AD}^2 = \text{AB}^2 + \text{BD}^2 = a^2(4-2\sqrt{2})$, so that $\text{AD} = a\sqrt{(4-2\sqrt{2})}$.

We thus get $\sin 22\frac{1}{2}^\circ = \frac{\sqrt{2}-1}{\sqrt{(4-2\sqrt{2})}} = \sqrt{\frac{3-2\sqrt{2}}{4-2\sqrt{2}}}$

$= \frac{\sqrt{(4-2\sqrt{2})}}{2\sqrt{2}}$, on simplification;

$$\cos 22\frac{1}{2}^\circ = \frac{1}{\sqrt{(4-2\sqrt{2})}} = \frac{\sqrt{(4+2\sqrt{2})}}{2\sqrt{2}};$$

$$\tan 22\frac{1}{2}^\circ = \sqrt{2}-1; \ \cot 22\frac{1}{2}^\circ = \sqrt{2}+1.$$

By means of the formulæ $\sin(180^\circ - \text{A}) = \sin \text{A}$, $\cos(180^\circ - \text{A}) = \cos \text{A}$, etc., the ratios of any obtuse angle may be obtained from those of the supplementary acute angle. We shall thus obtain results like the following:—

$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 120^\circ = -\frac{1}{2}$,

$\sin 135^\circ = \frac{1}{\sqrt{2}}$, $\cos 135^\circ = -\frac{1}{\sqrt{2}}$, $\sin 150^\circ = \frac{1}{2}$, $\cos 150^\circ = -\frac{\sqrt{3}}{2}$.

Again we shall have $\sin 180^\circ = 0$, $\cos 180^\circ = -1$,

$\sin 165^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\cos 165^\circ = -\frac{\sqrt{3}+1}{2\sqrt{2}}$.

These values are frequently required, and may be also obtained directly from the figures.

Powers of the trigonometrical ratios have often to be employed; *e. g.* $(\sin A)^2$, $(\cos B)^3$, $(\tan x)^{\frac{1}{2}}$. In the case of the *positive integral* powers, the indices are attached *to the abbreviations* of the ratios and put *before the angle.* Thus the first two powers above are written $\sin^2 A$, $\cos^3 B$.

Exercises

47. Write down the values of

$$\sin 90^\circ,\ \cos 180^\circ,\ \tan 150^\circ,\ \cot 135^\circ,\ \sec 120^\circ,\ \operatorname{cosec} 0^\circ.$$

48. If $\alpha = 60^\circ$, $\beta = 90^\circ$, $\gamma = 120^\circ$, prove that

$$\sin^2 \beta \sin \gamma + \cos^2 \beta \cos \gamma + \sin \alpha = \sqrt{3};$$

and find the value of $(\cot \alpha - \cot \beta)(\cot \beta - \cot \gamma)(\tan \gamma - \tan \alpha)$.

49. Prove the identities—

(1) $\cos^2 60^\circ + \cos 60^\circ = \cos^2 30^\circ$;
(2) $3 \sin 45^\circ - 4 \sin^3 45^\circ + \cos 135^\circ = 0$;
(3) $\tan 45^\circ - \tan 30^\circ = (2 - \sqrt{3})(1 + \tan 45^\circ \tan 30^\circ)$.

50. Find x from the equation

$$x(\cot^3 30^\circ - \tan 60^\circ - \tan 150^\circ) = 2 \sec 0^\circ + 5 \operatorname{cosec} 90^\circ.$$

*51. Show that the formula "area of triangle $= \frac{1}{2}$ (product of 2 sides) (sine of included angle)," holds good when this angle is right or obtuse.

52. ABC is a triangle right-angled at C; prove that

$$\cot \frac{A}{2} = \frac{AB + AC}{BC}.$$

53. The sine of an obtuse angle is $\frac{3}{5}$: draw the angle and measure its cosine and cotangent.

54. The ratio of the sine of an angle to its cotangent is $-\frac{1}{\sqrt{2}}$: find the angle.

55. Taking an angle of 135° and proceeding as in Exam. 2 solved above, prove that

$$\sin 67\tfrac{1}{2}^\circ = \frac{1}{\sqrt{(4 - 2\sqrt{2})}},\quad \cos 67\tfrac{1}{2}^\circ = \frac{\sqrt{(4 - 2\sqrt{2})}}{2\sqrt{2}}.$$

56. Write down the values of $\sin 112\frac{1}{2}^\circ$, $\cos 165^\circ$, $\tan 105^\circ$.

57. Find the perimeter of a regular octagon inscribed in a circle of radius $\sqrt{2}$, (i) by means of the Tables, (ii) by employing the values of the ratios of $22\frac{1}{2}^\circ$.

58. Find the perimeter of a regular figure of 12 sides described about a circle of radius 1·2 inches.

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59. By proceeding as in Exam. 2 or 3 above, show that

$$\tan 11\tfrac{1}{4}^\circ = \sqrt{(4 + 2\sqrt{2})} - \sqrt{2} - 1.$$

60. By means of Prop. XXII or otherwise, prove that

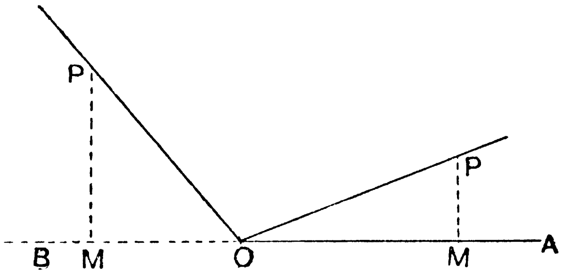
$$\tan 7\tfrac{1}{2}^\circ = \sqrt{6} + \sqrt{2} - 2 - \sqrt{3},$$

$$\text{and } \cot 7\tfrac{1}{2}^\circ = \sqrt{6} + \sqrt{2} + 2 + \sqrt{3}.$$

CHAPTER XII.

ALGEBRAICAL RELATIONS: IDENTITIES AND EQUATIONS.

§ 28. ALGEBRAICAL RELATIONS BETWEEN THE RATIOS OF THE SAME ANGLE: IDENTITIES.



Let AOP be any angle of magnitude θ, draw PM perp. to AOB. Then, by definition $\sin\theta = \frac{\text{MP}}{\text{OP}}$, $\operatorname{cosec}\theta = \frac{\text{OP}}{\text{MP}}$, whether θ is acute or obtuse;

$$\therefore \quad \sin\theta \times \operatorname{cosec}\theta = 1,$$

and
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sin\theta = \frac{1}{\operatorname{cosec}\theta}.$$

Similarly, the definitions give at once

$$\cos\theta \times \sec\theta = 1, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cos\theta = \frac{1}{\sec\theta};$$

$$\tan\theta \times \cot\theta = 1, \quad \cot\theta = \frac{1}{\tan\theta}, \quad \tan\theta = \frac{1}{\cot\theta}.$$

Again, in both cases,

$$\sin\theta \div \cos\theta = \frac{\text{MP}}{\text{OP}} \div \frac{\text{OM}}{\text{OP}} = \frac{\text{MP}}{\text{OM}} = \tan\theta,$$

$$\cos\theta \div \sin\theta = \frac{\text{OM}}{\text{OP}} \div \frac{\text{MP}}{\text{OP}} = \frac{\text{OM}}{\text{MP}} = \cot\theta;$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Lastly, from either right-angled triangle OPM, we have $\text{OP}^2 = \text{OM}^2 + \text{MP}^2$.

Dividing throughout by OP^2, we get

$$1 = \left(\frac{OM}{OP}\right)^2 + \left(\frac{MP}{OP}\right)^2 = \cos^2\theta + \sin^2\theta :$$

thus $$\sin^2\theta + \cos^2\theta = 1.$$

Dividing the same identity by OM^2 and MP^2 in turn, we get

$$\left(\frac{OP}{OM}\right)^2 = 1 + \left(\frac{MP}{OM}\right)^2, \text{ that is, } \mathbf{1 + \tan^2\theta = \sec^2\theta};$$

and $$\left(\frac{OP}{MP}\right)^2 = 1 + \left(\frac{OM}{MP}\right)^2, \text{ that is, } \mathbf{1 + \cot^2\theta = \operatorname{cosec}^2\theta}.$$

The student is advised to make himself thoroughly familiar with these *identical relations* They are not all independent, and may be shown to reduce to five. For instance, if the first three, one of the next two, and one of the last three, results are given, the remaining three can be deduced. For, if $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$, then evidently $\cot\theta = \frac{\cos\theta}{\sin\theta}$. Again, if $\sin^2\theta + \cos^2\theta = 1$, dividing out by $\sin^2\theta$ we get $1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$, or $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, as $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ is given and $\cot\theta = \frac{\cos\theta}{\sin\theta}$ is proved. So also, dividing out by $\cos\theta$, we get $1 + \tan^2\theta = \sec^2\theta$.

When the angle is obtuse, OM is negative; but OM^2 is of course positive, and the identity $OP^2 = OM^2 + MP^2$ holds good in all cases.

These relations are *identities*, that is, hold for *any value whatsoever* of the angle θ. They may be employed to prove more complex relations in the manner of the following examples.

Examples. 1. Prove the identity

$$\operatorname{cosec}^2 A - \sec^2 A = \cot^2 A - \tan^2 A.$$

$$\text{The left side} = (1 + \cot^2 A) - (1 + \tan^2 A)$$
$$= \cot^2 A - \tan^2 A = \text{the right side}.$$

Or we may proceed thus.

$$\text{The left side} = \frac{1}{\sin^2 A} - \frac{1}{\cos^2 A}:$$

$$\text{also the right side} = \frac{\cos^2 A}{\sin^2 A} - \frac{\sin^2 A}{\cos^2 A} = \frac{\cos^4 A - \sin^4 A}{\sin^2 A\cos^2 A}$$
$$= \frac{\cos^2 A - \sin^2 A}{\sin^2 A\cos^2 A}, \text{ as } \cos^2 A + \sin^2 A = 1,$$
$$= \frac{1}{\sin^2 A} - \frac{1}{\cos^2 A}. \text{ Thus the two sides are equal.}$$

This process is longer than that given above; but it has the advantage of reducing the other trigonometrical ratios of A to sinA and cosA only. The student will find this reduction very convenient in most cases.

2. Prove that

$$\frac{a+\sqrt{(a^2+b^2)}.\ \sin\theta}{b+\sqrt{(a^2+b^2)}.\ \cos\theta}+\frac{b-\sqrt{(a^2+b^2)}.\ \cos\theta}{a-\sqrt{(a^2+b^2)}.\ \sin\theta}=0.$$

Multiply out by the product of the denominators : the right-hand side remains zero; and the left-hand side becomes

$$a^2-(a^2+b^2)\sin^2\theta\ +b^2-(a^2+b^2)\cos^2\theta$$
$$=(a^2+b^2)-(a^2+b^2)(\sin^2\theta+\cos^2\theta)$$
$$=(a^2+b^2)-(a^2+b^2)=0,\ \text{as}\ \sin^2\theta+\cos^2\theta=1.$$

3. Prove that $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$.

We have $\sin^6\theta+\cos^6\theta$

$$=(\sin^2\theta+\cos^2\theta)(\sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta)$$
$$=\sin^4\theta+2\sin^2\theta\cos^2\theta+\cos^4\theta-3\sin^2\theta\cos^2\theta$$
$$=(\sin^2\theta+\cos^2\theta)^2-3\sin^2\theta\cos^2\theta$$
$$=1-3\sin^2\theta\cos^2\theta.$$

The following method of establishing the result may be noted. It is known that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$. Now $\sin^2\theta+\cos^2\theta+(-1)=0$, as we know; $\therefore(\sin^2\theta)^3+(\cos^2\theta)^3+(-1)^3=3(\sin^2\theta)(\cos^2\theta)(-1)$.

This gives $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$.

The student should similarly prove

$$\sec^6 A-\tan^6 A=1+3\sec^2 A\tan^2 A,$$

and $1+\cot^8 A+\operatorname{cosec}^8 A=2\cot^4 A+2\operatorname{cosec}^4 A+2\cot^4 A\operatorname{cosec}^4 A$.

We have seen that when any one of the trigonometric ratios of an acute or obtuse angle is given the angle is fixed, and by constructing it the other ratios can be obtained. By means of the formulæ this construction may be avoided, and the remaining ratios can be found algebraically.

4. Given $\sin A=\frac{12}{13}$; find the other functions of A which is an acute angle.

$\because \cos^2 A=1-\sin^2 A=1-\frac{144}{169}=\frac{25}{169}$, $\therefore \cos A=\pm\frac{5}{13}$.

As A is acute we take the positive value, $\cos A=\frac{5}{13}$.

Also $\tan A=\sin A\div\cos A=\frac{12}{5}$; $\cot A=1\div\tan A=\frac{5}{12}$,

$\sec A=\frac{13}{5}$, $\operatorname{cosec} A=\frac{13}{12}$.

If $\frac{12}{13}$ were given as the sine of an obtuse angle, the cosine would be $-\frac{5}{13}$; and we should have

$\tan A=-\frac{12}{5}$, $\cot A=-\frac{5}{13}$, $\sec A=-\frac{13}{5}$, $\operatorname{cosec} A=\frac{13}{12}$, the signs of the functions being determined as in § 26.

5. Given $\cot A = \dfrac{2ab}{a^2 - b^2}$, find the other functions of A, a and b being positive and $a < b$.

As the cotangent is negative, ∠ A is *obtuse*.

$$\because \quad \operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{4a^2b^2}{(a^2 - b^2)^2} = \frac{(a^2 + b^2)^2}{(a^2 - b^2)^2};$$

$$\therefore \quad \operatorname{cosec} A = \pm \frac{a^2 + b^2}{a^2 - b^2}.$$

As the cosecant of an obtuse angle is positive, we take the negative sign which gives a positive result on the whole.

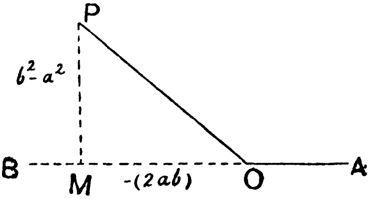
Thus $\operatorname{cosec} A = -\dfrac{a^2 + b^2}{a^2 - b^2} = \dfrac{a^2 + b^2}{b^2 - a^2}$, and $\therefore$ $\sin A = \dfrac{b^2 - a^2}{a^2 + b^2}$.

Again $\cot A = \dfrac{\cos A}{\sin A}$, so that $\cos A = \cot A . \sin A$;

$$\therefore \cos A = \frac{2ab}{a^2 - b^2} \cdot \frac{b^2 - a^2}{a^2 + b^2} = -\frac{2ab}{a^2 + b^2};$$

and $\therefore$ $\sec A = -\dfrac{a^2 + b^2}{2ab}$. Of course $\tan A = \dfrac{a^2 - b^2}{2ab}$.

In such examples it is often convenient to draw a rough figure for the angle, and proceed to employ the definitions after finding such lengths as are necessary. In the present case, let AOP be the obtuse angle with the given cotangent. Draw PM perp. to AOB, and suppose $OM = -2ab$, $MP = b^2 - a^2$: for $2ab$ and $b^2 - a^2$ are positive, and the abscissa OM is negative while the ordinate MP is positive.



Now $OP^2 = OM^2 + MP^2 = 4a^2b^2 + (b^2 - a^2)^2 = (a^2 + b^2)^2$;

$\therefore$ $OP = a^2 + b^2$, as the revolving length is always positive.

We now read off the other ratios :—

$$\sin A = \frac{b^2 - a^2}{a^2 + b^2}, \quad \cos A = -\frac{2ab}{a^2 + b^2}, \quad \tan A = -\frac{b^2 - a^2}{2ab},$$

$$\sec A = -\frac{a^2 + b^2}{2ab}, \quad \operatorname{cosec} A = \frac{a^2 + b^2}{b^2 - a^2}.$$

If no relation like $a < b$ is given, we cannot determine the sign of cot A; in this case, the ambiguity in the value of cosec A cannot be removed.

Exercises.

61. If the cosecant of an angle is $\frac{2}{\sqrt{3}}$, what is the secant? And if the secant is 2, what is the cosecant?

62. Find the cotangent of $22^\circ 30'$ by bisecting one acute angle of an isosceles right-angled triangle; and hence by means of formulæ find the other ratios of the angle.

*63. Given $\sin A = s$, find the other functions of A in terms of s.

*64. Given $1 - \cos A = v$, find all the trigonometrical ratios of A.

The defect of the cosine of an angle from unity is called the *versed sine* of the angle. Thus the quantity v given above is the versed sine of A, which is abbreviated into *vers*A.

65. Find an angle θ such that

$$\text{(i) } \sin\theta = 1 + \cos\theta, \quad \text{(ii) } \operatorname{cosec}\theta = 1 + \cot\theta.$$

66. Simplify the following expressions:—

$$\frac{\sec\theta\cot\theta - \operatorname{cosec}\theta\tan\theta}{\cos\theta - \sin\theta}; \quad \operatorname{vers}^2 A - 2\operatorname{vers} A;$$

$$(7\cos\theta + 6\sin\theta - 3)^2 + (7\sin\theta - 6\cos\theta + 3)^2 - 94;$$

$$\left\{\frac{\operatorname{cosec}\theta + \sec\theta}{\operatorname{cosec}\theta - \sec\theta} - \frac{\operatorname{cosec}\theta - \sec\theta}{\operatorname{cosec}\theta + \sec\theta}\right\}\operatorname{cosec}\theta\sec\theta.$$

67. In a triangle ABC right-angled at B, $CB = 2m$, $AB = m^2 - 1$: find sin BAC, cot BAC, sec BCA, cosec BCA.

68. Given $\cos\theta = \frac{a(a+2b)}{a(a+2b) + 2b^2}$, find $\cot\theta$ and $\operatorname{cosec}\theta$.

*69. Express all the other trigonometrical ratios of an angle in terms of (i) the tangent, (ii) the secant, of the angle.

70. If $\tan\theta = \frac{2}{3}$, prove that

$$\frac{5}{3}\sin\theta - \cos\theta = \frac{1}{13}\operatorname{cosec}\theta - \frac{1}{26}\sec\theta.$$

71. Simplify $(2\tan A - \sec A + 1)(2\tan A + \sec A + 1) - (\tan A - 2\sec A + 2)(\tan A + 2\sec A + 2)$; and prove that

$$\sqrt{\left\{\frac{1 + \tan A}{1 - \tan A}\right\}} + \sqrt{\left\{\frac{1 - \tan A}{1 + \tan A}\right\}} = \frac{2\cos A}{\sqrt{(2\cos^2 A - 1)}}.$$

72. If $\cot\theta = \frac{a}{b}$, find the values of

$$a\sin\theta - b\cos\theta, \quad \frac{a\cos\theta - b\sin\theta}{a\sin\theta + b\cos\theta}, \quad \sin\theta\,(1 + \cos^2\theta).$$

Establish the identities in the following examples (73 − 85) :—

73. $\sin A(1+\tan A)+\cos A\,(1+\cot A)=\sec A+\operatorname{cosec} A.$

74. $(\sin A-\sin^3 A)^2+(\cos^3 A-\cos A)^2=\sin^2 A\cos^2 A.$

75. $\cos^6\beta-\sin^6\beta=(\cos^2\beta-\sin^2\beta)(1-\cos\beta\sin\beta)(1+\cos\beta\sin\beta).$

76. $2\,(\cos^6\beta+\sin^6\beta)-3\,(\cos^4\beta+\sin^4\beta)+1=0.$

77. $4\,(\cos^6\beta+\sin^6\beta)=1+3\,(\cos^2\beta-\sin^2\beta)^2.$

78. $\cos^8\beta-\sin^8\beta=(\cos^2\beta-\sin^2\beta)(1-2\cos^2\beta\sin^2\beta).$

79. $\sec^8\beta+\tan^8\beta+1=2\sec^4\beta+2\tan^4\beta+2\sec^4\beta\tan^4\beta.$

80. $(1+\tan\gamma+\cot\gamma)(\sin\gamma-\cos\gamma)=\sin^2\gamma\sec\gamma-\cos^2\gamma\operatorname{cosec}\gamma.$

81. $\dfrac{(1+\cot A)^2-(1-\cot A)^2}{(1+\cot A)^2+(1-\cot A)^2}=(\sin A+\cos A)^2-1.$

82. $\dfrac{\sin A\;(\cos A-\sin A)}{1-2\sin^2 A}=\dfrac{\tan A}{1+\tan A}.$

83. $\dfrac{\cos^2 A+\sin A-\sin^2 A}{\cos A\;(1+2\sin A)}=\dfrac{\cos A}{1+\sin A}.$

84. $\dfrac{\sin^2 A+7\cos A-7\cos^2 A}{\sin A\;(1+8\cos A)}=\dfrac{\sin A}{1+\cos A}.$

85. $\left\{\dfrac{\sec A+\tan A+1}{\sec A-\tan A+1}\right\}^2=\dfrac{\operatorname{cosec} A+1}{\operatorname{cosec} A-1}.$

§ 29. TRIGONOMETRICAL EQUATIONS.

An equality which is true for all values of the unknown quantities (variables) involved in it is called an *identical equation,* or briefly an *identity.* Thus $x^2-4=(x+2)(x-2)$, $1+\tan^2\theta=\sec^2\theta$, are *identities* as they hold for any values whatsoever of x, θ.

An equality which is true for only one value or a definite number of values of the variables involved may be called a *conditional equation*; more usually, it is said to be simply an *equation.* Thus $x^2-1=2x+2$ is an *equation* and not identity, as it holds good only when $x=-1$ or $x=3$. Similarly, $\sin\theta=\sqrt{2}-\cos\theta$ is found to hold good when $\theta=45^0$ and for no other acute or obtuse value of θ; it is therefore an equation.

If an equation involves only one of the trigonometrical ratios of an unknown angle, it is solved by finding algebraically the value of that function, and then determining the angle by means of the Tables. If the ratio has one of the simple values found in § 27, the corresponding angle may be written down at once.

If several trigonometrical ratios of an unknown angle occur, the equation is solved by expressing the other ratios in terms of *some one* of them, say the sine or the tangent; we can then find the value of this ratio algebraically. On account of this substitution, the ratio to be determined will often occur in expressions under a square root sign; in this case we can employ the usual process of *rationalizing* or *transposing and squaring*. All roots found thus will not, however, necessarily satisfy the original equation, and we shall have to determine the proper roots by verification. (See Hall and Knight, *Elem. Algebra*, § 281 Ex. 3 and Note.)

As $\cos A = \sqrt{(1-\sin^2 A)}$, $\tan A = \sin A \div \sqrt{(1-\sin^2 A)}$, it is seen that $\cos A$ and $\tan A$ involve quadratic surds when $\sin A$ is given. Thus if $\sin A$, $\cos A$, $\tan A$ occur together in an equation, the equation in fact is a surd equation in $\sin A$. Similarly we may reason when $\tan A$ and $\sec A$ occur together.

It is important to note the following results in connection with the solution of equations :—

The *sine* and *cosine* of an angle *cannot be numerically greater than unity,* which is the maximum value of either.

The *secant* and *cosecant* of an angle *cannot be numerically less than unity,* which is the minimum value of either.

The *tangent* and *cotangent* of an angle *can have any values* from 0 to ∞, positive or negative.

These results follow at once from the rule about the signs of ordinates and abscissæ, and from the fact that in a right-angled triangle the hypotenuse cannot be less than either side while one side may bear to the other any ratio whatever.

Examples. 1. Solve $\sin 3x = \frac{1}{2}$.

The angles of 30° and 150° have $\frac{1}{2}$ for their sine;

$$\therefore\ 3x = 30^\circ \text{ or } 150^\circ, \text{ so that } x = 10^\circ \text{ or } 50^\circ.$$

2. $15\cos^2 x - 19\cos x + 6 = 0$.

Factorizing the left side, $(3\cos x - 2)(5\cos x - 3) = 0$;

$$\text{hence either } \cos x = \frac{2}{3} \text{ or } \cos x = \frac{3}{5}.$$

$$\therefore\ x = 48^\circ\ 11' \text{ or } 53^\circ\ 9', \text{ from the Tables.}$$

When the factors are not rational, the method of *completing the square* may be employed.

$12 \tan^2 \theta + 19 \sec \theta = 6.$

As $\tan^2\theta = \sec^2\theta - 1$, we get $12 \sec^2\theta + 19 \sec \theta - 18 = 0.$

$\therefore \quad (3 \sec \theta - 2)(4 \sec\theta + 9) = 0,$

and $\sec\theta = \frac{2}{3}$ or $-\frac{9}{4}$.

The secant is never less than unity ; hence we reject the first solution.

The second gives $\sec \theta = -2\cdot25$.

Now $2\ 25$ is the secant of $63^\circ37'$, and $\sec(180^\circ - \theta) = -\sec \theta$;

$\therefore \quad \theta = 180^\circ - 63^\circ37' = 116^\circ23'.$

4. $\sqrt{2}.\ \sin\beta + 2\sqrt{2}.\ \cos\beta \quad 3.$

$\therefore \quad 2\sqrt{2} \cos \beta = 3 - \sqrt{2} \sin \beta$;

squaring, $8(1 - \sin^2 \beta) = 9 - 6\sqrt{2} \sin \beta + 2 \sin^2\beta,$

or $10 \sin^2\beta - 6\sqrt{2} \sin\beta + 1 = 0.$

This gives $\sin \beta = \frac{1}{\sqrt{2}}$ or $\frac{1}{5\sqrt{2}}$.

From the first, $\beta = 45^\circ$ or 135° ;

from the second, $\beta = 8^\circ\ 8'$ or $171^\circ52'$.

On substitution in the original equation, it is found that *both* obtuse values are *inapplicable* ,

$\therefore \quad \beta = 45^\circ$, or $8^\circ\ 8'$.

In such examples it is preferable to get an equation for the *cosine*, as there is only one angle between 0° and 180° with a given positive or negative cosine.

5 $\sqrt{3} \sin\beta + \cos\beta = \sqrt{3}.$

Transpose $\cos \beta$ and square ;

$3(1 - \cos^2\beta) = 3 - 2\sqrt{3} \cos \beta + \cos^2 \beta,$

$\therefore \quad 4 \cos^2\beta - 2\sqrt{3} \cos \beta = 0.$

Hence $\cos\beta = 0$ or $\frac{\sqrt{3}}{2}$, so that $\beta = 90^\circ$ or 30°.

Both values are applicable.

6. $1 + \sin^2\beta = 3 \sin \beta \cos \beta.$

Divide throughout by $\cos^2 \beta$;

$\sec^2 \beta + \tan^2\beta = 3 \tan \beta, \ \therefore \ 2 \tan^2\beta - 3 \tan\beta + 1 = 0.$

Hence $\tan \beta = 1$ or $\frac{1}{2}$;

$\therefore \beta = 45^\circ$, or $26^\circ\ 34'$ from the Tables.

When the terms involving the ratios of the unknown angle are homogeneous in the sine and cosine, it is advantageous to divide out by the highest power of one or both of these ratios ; the equation will generally be reduced to terms in powers of the tangent or cotangent as in the example above. Otherwise, squaring will be required, and this would introduce extrinsic roots as we have seen.

7. If $\cot\beta\ (1+\sin\beta) = 4m$, $\cot\beta\ (1-\sin\beta) = 4n$, find $\cot\beta$ and $\sin\beta$, and show that

$$(m^2-n^2)^2 = mn.$$

By division $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{m}{n}$,

so that $\sin\beta = \dfrac{m-n}{m+n}$.

Substituting this value in either equation, we obtain

$$\cot\beta = 2\,(m+n).$$

Now $\operatorname{cosec}\beta = \dfrac{1}{\sin\beta} = \dfrac{m+n}{m-n}$;

and it is known that $\operatorname{cosec}^2\beta = 1+\cot^2\beta$;

$$\therefore \quad \left(\frac{m+n}{m-n}\right)^2 - 1 = 4\,(m+n)^2,$$

which gives $mn = (m^2-n^2)^2$.

As β is supposed to be the *same* in *both* equations, the values of β found from each must be the same; on equating these a relation between the coefficients will result. Such a relation is called the *eliminant* of the two given equations, and may be found indirectly without actually solving each for β. When one ratio of an angle is known we have seen that the others could be determined; hence we may find two ratios from the two equations, and from the relation between them obtain the eliminant.

Exercises.

86. Obtain the values of A and B from the following pairs of equations:

(i) $\sin(A+B) = \frac{1}{2}$, $\cos(A-B) = \frac{\sqrt{3}}{2}$;

(ii) $\cos(A+B) = -\frac{1}{2}$, $\tan(A-B) = \sqrt{3}$.

87. By means of the Tables find the angle x in the following cases:—

(i) $\sin 2x = {\cdot}945$; (ii) $\cos(x+60^\circ) = {\cdot}145$;

(iii) $\tan 3x = 2{\cdot}459$; (iv) $2\cot x + 1{\cdot}514 = 0$.

Solve the equations in the following examples (88–96):—

88. $\cos^2\theta - 2\sin\theta + \frac{1}{4} = 0$.

89. $\sin^3\theta - \cos^3\theta = 0$.

90. $2(\tan\theta + \cot\theta) = 5$.

91. $\sin x - \cos x = \sqrt{2}$.

92. $\sqrt{3}. \sin x + \cos x = 2$.

93. $2 \cos x + 1 = \cot x + 2 \sin x$.

94. $16 \sin^2 x \cos^2 x + 4 \sin^2 x - 5 = 0$.

95. $\dfrac{1+\tan x}{1-\tan x} = \dfrac{3(1-\tan x)}{1+\tan x}$

96. $\sqrt{2}. \sin^2 x + \cot x = \sqrt{2}$.

97. Solve the following equations:—

(i) $\tan x + 4 \cos^2 x = 3$;

(ii) $\sec^2 x + 3 \operatorname{cosec}^2 x = 8$;

(iii) $1 - 11 \cos^2 x = 3 \sin x \cos x$.

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98. Find cos A when

(i) $\tan A + \sec A = \frac{4}{3}$,

(ii) $\operatorname{cosec} A - \cot A = \frac{1}{4}$;

and verify the solutions.

99. Eliminate δ from the following pairs of equations:—

(i) $x \cos \delta + y \sin \delta = a, \quad x \sin \delta - y \cos \delta = b$;

(ii) $a \cos \delta + b \sin \delta = c = a' \cos \delta + b' \sin \delta$;

(iii) $\operatorname{cosec} \delta - \sin \delta = m, \quad \sec \delta - \cos \delta = n$.

100. Eliminate a, b, c from the following equations, and obtain a relation between the trigonometrical functions of A, B, C:—

$b \cos C + c \cos B = a, \quad c \cos A + a \cos C = b, \quad a \cos B + b \cos A = c$.

CHAPTER XIII.

TRIGONOMETRIC AND LOGARITHMIC TABLES.

§ 30. USE OF TRIGONOMETRIC TABLES: SIMPLE PROBLEMS.

In all practical problems of Trigonometry the student will find that some ratios of given angles are required; and conversely, that angles have to be found from given values of some of their ratios. To effect this tables are constructed giving the sine, the tangent and the secant of all angles below 90° at intervals of 1′; these are respectively called *Tables* of *Natural Sines, Natural Tangents* and *Natural Secants.* They give the values of the ratios to four places of decimals and are used as explained below.

The degrees in the given angle are to be taken from the left-hand column of any page of the Tables; and the minutes, at intervals of six (0′, 6′, 12′,...), from the line at the top. At the intersection of the line and the column through these points will be found the value of the required ratio. If an angle, like 25° 32′, does not contain an exact multiple of 6′, we first find the ratio for 25° 30′ which is the next lower angle containing such a multiple; we then look to the last five columns, headed 1′, 2′,...5′, in the line of the degrees, and take the number given under 2′, and add it to the ratio found. Thus will be obtained the required ratio of such an angle. The number so added is often called the *correction* for the excess of minutes of the given angle over the angle whose ratio is first found.

In the Table of *natural sines* (and of cosines) the decimal points are all omitted, as no entry could be greater than unity. In the other Tables the decimal points and the integers preceding them are printed *only in the first column* of the values of the ratios. No entries for obtuse angles are given; for any ratios of such angles may be reduced to corresponding ratios of their supplements (§ 26).

It will be seen that the *sine*, *tangent* and *secant* increase for an acute angle as the angle increases; hence for an *excess* of the angle the correction made should be *additive*. The *cosine*, *cotangent* and *cosecant* decrease as the angle increases; hence, if separate tables of them are employed, for an *excess* of the angle the correction made would be *subtractive*. But separate tables of these ratios are not necessary, as they are respectively the sine, tangent and secant of the *complementary* angles.

Examples. 1. Find $\sin 55^\circ 26'$ and $\cos 31^\circ 49'$.

From the Table of natural sines we get
$\sin 55^\circ 24' = \cdot 8231$;
adding for increase of $2'$ $\quad 3$,
we have $\sin 55^\circ 26' = \cdot 8234$.
As $\cos 31^\circ 49' = \sin 58^\circ 11'$,
we take out $\sin 58^\circ 6' = \cdot 8490$,
and add for increase of $5'$ $\quad 8$;

$\therefore \cos 31^\circ 49' = \sin 58^\circ 11' = \cdot 8498$.

If we use the Table of cosines, we get
$\cos 31^\circ 48' = \cdot 8499$;
subtracting for increase of $1'$ $\quad 2$,
we have $\cos 31^\circ 49' = \cdot 8497$.

The difference of 1 in the last place is due to the fact that the Tables give the *nearest* value of the *fourth* digit: more accurately, we should first work to five digits and then keep four.

2. Find $\tan 66^\circ 16'$ and $\cot 70^\circ 2'$.

From the Table of natural tangents
$\tan 66^\circ 12' = 2\cdot 2673$;
add for increase of $4'$ $\quad 74$;
thus $\tan 66^\circ 16' = 2\cdot 2747$.

And $\cot 70^\circ 2' = \tan 19^\circ 58'$;
now $\tan 19^\circ 54' = \cdot 3620$,
and increment for $4' = 13$;
$\therefore \tan 19^\circ 58' = \cot 70^\circ 2' = \cdot 3633$.

3. Find $\sec 37^\circ 1'$ and $\operatorname{cosec} 47^\circ 2'$.

As before $\sec(37^\circ + 1') = 1\cdot 2521 + \cdot 0003 = 1\cdot 2524$;
and $\operatorname{cosec} 47^\circ 2' = \sec 32^\circ 58'$
$= \sec(32^\circ 54' + 4') = 1\cdot 1910 + \cdot 0009 = 1\cdot 1919$.

When tables of secants are not available, the tables of sines may be used; the reciprocal of any sine will give the cosecant of the angle, and the reciprocal of the sine of the complement will give the secant of the angle.

When the sine or cosine, tangent, . . . of an angle is given, the angle may be found from these Tables as in the following examples:—

4. Find the angle whose sine is ·8342 and that whose cosine is ·7064.

From the Table of natural sines

$$\begin{array}{lrr} & \cdot 8339 = \sin & 56^\circ\ 30'; \\ \text{add for increase of} & 3 & 2'; \\ \therefore\ \text{angle required} = & & 56^\circ\ 32'. \end{array}$$

$$\begin{array}{lrr} \text{From the same Table} & \cdot 7059 = \sin & 44^\circ\ 54'; \\ \text{add for increase of} & 5 & 2\tfrac{1}{2}'; \\ \therefore & \cdot 7064 = \sin & 44^\circ\ 56\tfrac{1}{2}' = \cos\ 45^\circ\ 3\tfrac{1}{2}'. \end{array}$$

$\therefore$ angle required $= 45^\circ\, 3\tfrac{1}{2}'$.

In the second case the increment 5 is not found in any of the small columns headed 1′, 2′, . . 5′. It is midway between 4 and 6 which are the increments for 2′ and 3′; hence we take the increment of the angle to be $2\tfrac{1}{2}'$. It is usual to take it as 3′, the angles being generally estimated to the nearest minute.

5. Find the angle whose cotangent is ·84.

The immediately lower number in the Table of tangents is $\cdot 8391 = \tan 40^\circ$; for an increment of 9 (in the fourth place) we add 2′ to the angle.

Thus $84 = \tan 40^\circ 2' = \cot 49^\circ 58'$.

Here the increment lies between the printed increments 5 and 10, but is much nearer the second; hence we take the corresponding increment of the angle to be 2′.

6. Find the acute value of α satisfying the equation

$$7 \tan^2 \alpha + 8\sqrt{3} \tan \alpha = 1.$$

Solving by the ordinary rule and taking the *positive* sign, we have

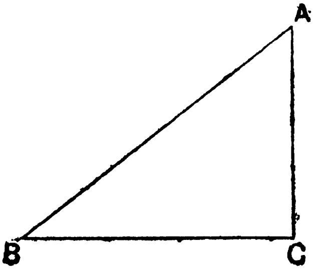
$$\tan \alpha = \frac{-4\sqrt{3} + \sqrt{55}}{7} = \frac{\cdot 488}{7} = \cdot 0697.$$

Hence we get from the Tables $\alpha = 3^\circ 54' + 5' = 3^\circ 59'$.

The negative sign will give $\alpha = 116^\circ\ 1'$.

When one angle of a triangle is right, if we know any two of its sides or any one side and one acute angle, the remaining sides and angles may be readily found by geometrical construction. The process of finding their values numerically from the definitions of the trigonometrical ratios of the angles and their values furnished by the Tables, is called the *solution* of the triangle. For instance if the $\triangle$ ABC has a right angle at C, and if we know the length BC and the $\angle$ B, we can find the remaining sides and angles thus :— $\angle A = 90^\circ - \angle B$; and

$\because \dfrac{CA}{BC} = \tan B$, $\therefore\ CA = BC \tan B$;



Each of the three sides and three angles of a triangle is called a *part* or *element* of the triangle; and it is seen without difficulty that a right-angled triangle can be solved when two of its parts (including a side), besides the right angle, are known.

Many objects on the earth are practically inaccessible; in order to determine their distance from some known point, we should measure *one length* at least between two known points and *two angles* at least between lines of known directions. Thus, in the figure given above, A being the inaccessible top of an object whose height is CA, we measure the length CB and the $\angle$B; the $\angle$C being given (90°), three *parts* of the $\triangle$ABC are known, and the other parts may be calculated. By means of suitable instruments angles can be measured with great accuracy, but it is always difficult to measure lengths. Hence the calculations are made to depend on a single length or on as few lengths as possible.

When an object is above the level of the observer's eye, the angle made with the horizontal plane through the eye by the line joining the object to the eye is called the *angle of elevation* of the object, or, briefly, its *elevation* or *altitude*. When the object is below the level of the eye, the corresponding angle is called the *angle of depression* of the object, or, simply, its *depression*.

Examples (*Contd.*). 7. In a $\triangle$ABC, right-angled at C, CA = 50, $\angle$ABC = 34°12′: find the other parts of the $\triangle$.

$\because \angle A + \angle B = 90^\circ, \quad \therefore \angle A = 90^\circ - 34^\circ\,12' = 55^\circ\,48'.$

Again, $BC \div CA = \tan A$; $\therefore BC = CA \tan A$

$$= 50 \times 1{\cdot}4715 = 73{\cdot}575;$$

and $AB = CA \sec A = 50 \times 1{\cdot}7791 = 88{\cdot}955.$

If the triangle is drawn with the side CA vertical, the question is seen to be the same as the following:—from the top of a tower 50 ft. high the depression of an object in the horizontal plane through its base is found to be 34° 12′; find the distances of the object from the top and foot of the tower By *top* and *foot* we usually mean the *highest point* of the tower and the *point vertically below* this point on the horizontal base.

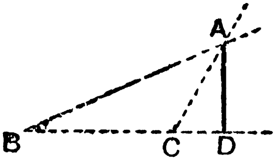
8. At a point on level ground 70 yards from the base of a tower the elevation of the tower is found to be 60°: find the height of the tower.

With the figure of the preceding example, we have here BC = 70 and $\angle$ABC = 60°, $\angle$ ACB being right.

$\therefore CA = BC \tan 60^\circ = 70\sqrt{3} = 121{\cdot}24.$

Thus the height is nearly $121\frac{1}{4}$ yards.

9. The base of a triangle is one metre, and the angles at the base are 121°, 25°: find the height of the triangle.



To any suitable scale draw BC = 10 (decim.); make ∠ ABC = 25°, ∠ ACB = 121°; and draw AD perp. to BC.

$\therefore \frac{BD}{DA} = \cot B$, $\therefore BD = DA \cot 25°$;

so also $CD = DA \cot ACD = DA \cot 59°$.

$\therefore BD - CD$, *i. e.* $BC = DA(\cot 25° - \cot 59°)$,

$$\text{so that } DA = \frac{10}{\cot 25° - \cot 59°} = \frac{10}{2{\cdot}1445 - {\cdot}6009}$$

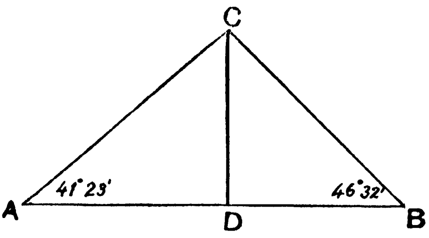
$$= \frac{10}{1{\cdot}5436} = 6{\cdot}478.$$

Thus the height is nearly 6·5 decimetres.

The solution should be verified by reading off AD to the scale on which BC is drawn. The result thus found is not so accurate as that obtained by calculation; but it forms a useful check on the latter, and may be conveniently employed when only a rough approximation to the height is required.

We can also find AB, AC, as they are respectively AD cosec 25°, AD cosec 59°. Thus we see that an oblique-angled triangle, ABC, can be *solved* when three of its *parts* (always *including one side*) are given, by taking it as the *difference* of two right-angled triangles, ABD and ACD. When the base angles are both acute, as in the following example, the triangle can be solved by dividing it into the *sum* of two right-angled triangles.

10. From two points A and B, 1500 yds. apart, an aeroplane C is observed to have simultaneous elevations of 41° 23′ and 46° 32′ respectively: if the plane ABC is vertical and the vertical line through C is between A and B, find the height of the aeroplane.



Let CD, the vertical line through C, meet the horizontal line AB in D.

As in the previous example,

$AD = DC \cot 41° 23′$,

and $BD = DC \cot 46° 32′$;

$\therefore$ DC (cot 41° 23′ + cot 46° 32′) = AD + BD = AB,

or DC (1·1350 + ·9479) = 1500, from the Tables.

$$\therefore DC = \frac{1500}{2{\cdot}0829} = \frac{15000000}{20829} = 720 \text{ yds., almost exactly.}$$

Exercises.

101. Find from the Tables—

sin 40° 25′, cos 70° 32′, cot 80° 52′, sec 33° 36′.

102. Obtain the value of A from—

$\cos A = \cdot 91$, $\tan A = -1\cdot 023$, $\sec A = 1\cdot 925$.

103. By means of the Tables solve the equations—

$4\sin^2\theta - 10\sin\theta + 5 = 0$ and $2\tan^2\theta + 5 = 8\sec\theta$.

104. In a triangle ABC, right-angled at C, find the other parts when—

(i) $c=47$, $a=24$, (ii) $a=45$, $b=25$,
(iii) $c=110$, $\angle A = 25^\circ 40'$, (iv) $b=10$, $\angle B = 70^\circ 25'$.

[If ABC is *any* triangle, the angles BAC, CBA, ACB are denoted respectively by A, B, C; and the sides BC, CA, AB, opposite to these angles, by a, b, c respectively.]

105. Find the side of a regular pentagon inscribed in a circle of radius 25 mm.

106. Find the radii of the inscribed and circumscribed circles of a regular dodecagon of side 2″.

107. In a triangle ABC, BC $= a$ and the perpendicular from A on BC $= p$: prove that

$$a = p\,(\cot B + \cot C)$$

when B and C are acute ; and examine the case when either B or C is obtuse.

108. At two points A and B the elevations of P, the top of an object whose base is in a line with AB, are α and β $(\alpha < \beta)$: if AB $= a$, prove that

$$AP = \frac{a \operatorname{cosec} \alpha}{\cot \alpha + \cot \beta} \text{ or } \frac{a \operatorname{cosec} \alpha}{\cot \alpha - \cot \beta}.$$

109. At two points, 100 yds. apart, on the bank of a straight river, the direction of an object on the opposite bank is observed to make angles of 30° and 70° with the first bank : find the breadth of the river, and the distance of the object from the first point of observation.

110. A tower 75 ft. high stands on a hillock; from the top and base of the tower the angles of depression of an object on level ground are found to be 45° and 30° respectively : find the distance of the object from the base of the tower.

111. A column 48 ft. high stands at a distance of 30 ft. from the bank of a river; when the altitude of the sun is 20° the shadow, falling right across the river, just reaches the other bank. Find the breadth of the river.

112. An observer standing 100 ft. from a pole observes the elevation of its top to be 11°: find the height of the pole, the observer's eye being $5\frac{1}{2}$ ft. above the ground. Find also how much farther the observer must go to see the pole at an elevation of $5\frac{1}{2}^\circ$.

113. A ladder a ft. long rests against a wall and makes an angle of 30° with it; the foot of the ladder is moved farther away from the wall and the angle then made with the wall is 75°. Find through what distances the top and foot of the ladder have moved.

114. From a point $40\frac{2}{5}$ ft. above the ground a column is observed to subtend an angle of 45°, and the distance of the foot of the column from the vertical line through the point of observation is found to be 100 ft. Find the height of the column.

115. A and B are two points on a straight river-bank, and T is the stem of a tree on the opposite bank; the angles TAB, TBA are found to be 45° 46′, 62° 31′ respectively, and AB is 22 yards. Find the breadth of the river at the point T.

116. The coast-line between two places A and B runs straight 15° E of N, and the bearings of a rock R from them are respectively 48° W of N and 12° S of W: find AR and the shortest distance of the rock from the coast, taking $AB = d$.

117. From a window in a house on one side of a street 50 ft. wide, the elevation of the top of the opposite house is found to be 32° and the depression of the base to be 28°. Find the heights of the opposite house and the point of observation.

118. The line joining two inaccessible objects, P and Q, is known to be at right angles to a base-line AB of length m metres; it is found that $\angle PAB = 60°$, $\angle PBA = 51°$ and $\angle QAB = 48° 12'$. Find the distance PQ.

§ 31. LOGARITHMS AND LOGARITHMIC TABLES.

The student will find that there is often a large amount of numerical computation required in the solution of practical problems. The labour of making the necessary calculations can be greatly reduced by referring to *logarithmic tables*, with which the student is already familiar to some extent. (See Hall and Knight, *Elem. Algebra*, Ch. XXXIX.) A brief recapitulation of the properties and uses of logarithms is given below.

Def. The **logarithm** of a number to a given **base** is the **index** of that power of the base which is equal to the number.

If N is the number, b the base and l the logarithm, we must have $b^l = N$; this is written thus—

$$l = \log_b N.$$

It should be noted that $b^{\log_b N} = N$, by definition; that the logarithm of the base is unity, as $b^1 = b$; and that the logarithm of unity is zero, as $b^0 = 1$.

Illustrations. $\because 2^4 = 16, \quad \therefore \log_2 16 = 4;$
so also $\log_3 243 = 5, \quad \log_{10} 100 = 2.$
Again, $\because 3^{-2} = \frac{1}{9}, \quad \therefore \log_3 (\frac{1}{9}) = -2;$
so also $\log_{10} (\cdot 001) = -3, \quad \log_6 (\cdot 1\dot{6}) = -1.$

The following are the chief properties of all systems of logarithms:—

(i) The logarithm of a product = the sum of the logarithms of the factors.

If $\log_b \mathrm{M} = x$, $\log_b \mathrm{N} = y$, then $b^x = \mathrm{M}$, $b^y = \mathrm{N}$;

$\therefore \mathrm{MN} = b^x . b^y = b^{x+y}$, that is—

$$\log_b (\mathrm{MN}) = x + y = \log_b \mathrm{M} + \log_b \mathrm{N}.$$

(ii) The logarithm of a quotient = the logarithm of the dividend diminished by the logarithm of the divisor.

Proceeding as above, we have

$$\frac{\mathrm{M}}{\mathrm{N}} = \frac{b^x}{b^y} = b^{x-y}, \text{ by the rules of indices;}$$

$$\therefore \quad \log_b \left(\frac{\mathrm{M}}{\mathrm{N}}\right) = x - y = \log_b \mathrm{M} - \log_b \mathrm{N}.$$

(iii) The logarithm of any power of a number = the logarithm of the number multiplied by the index of the power.

For $(b^x)^y = b^{xy}$, for all values of x and y; if $b^x = \mathrm{N}$, $\mathrm{N}^y = b^{xy}$.

$\therefore \quad \log_b(\mathrm{N}^y) = xy = (\log_b \mathrm{N}).\ y.$

We conclude from these results that logarithmic tables enable us to replace multiplication and division by addition and subtraction, and extraction of roots and raising to powers by division and multiplication.

It is found that the logarithms of numbers to a base are incommensurable quantities, except when the numbers happen to be rational powers of the base. Thus with the base 10, such numbers only as $100 = 10^2$, $\frac{1}{10} = 10^{-1}$, $\sqrt[3]{10}$, have logarithms which are definite rational numbers; the logarithms of the great majority of ordinary numbers are irrational, and require for their expression an interminable non-recurring decimal fraction preceded or not by an integer.

Def. The *positive fractional* part of a logarithm (expressed decimally) is called the *mantissa* of the logarithm, and the integral part (positive, zero or negative) is called its *characteristic.*

Any positive number greater than unity could be employed as the base of a logarithmic system; in practical calculations, however, the base is always **ten**, which is the radix or base of our decimal system of notation. Logarithms to the base 10 are called *common* logarithms. The common or ordinary system of logarithms possesses the two additional properties given below:—

(iv) In common logarithms, numbers having the same sequence of significant digits have the same mantissa. Thus,

$$\text{as } \log 1{\cdot}353 = {\cdot}1313,$$
$$\therefore \log (135{\cdot}3) = \log (1{\cdot}353 \times 10^2) = \log 1{\cdot}353 + \log 10^2$$
$$= {\cdot}1313 + 2 = 2{\cdot}1313;$$
$$\text{and} \log({\cdot}001353) = \log (1{\cdot}353 \div 10^3) = \log 1{\cdot}353 - \log 10^3$$
$$= {\cdot}1313 - 3.$$

In the last case the characteristic is not added to the mantissa, as then the latter would become negative; but the result is written thus—

$$\log {\cdot}001353 = \bar{3}{\cdot}1313.$$

So also $\log 135300 = 5{\cdot}1313$, $\log {\cdot}1353 = \bar{1}{\cdot}1313$.

(v) In common logarithms, the characteristic of the logarithm of any number may be found by inspection.

Thus, as $1{\cdot}353 > 10^0$ ($= 1$), but $< 10^1$,

$\therefore$ the characteristic of $\log 1{\cdot}353$ is zero, as this logarithm is only a positive proper fraction.

And $\because$ $135{\cdot}3 = 1{\cdot}353 \times 10^2$,

$\therefore$ $\log 135{\cdot}3 = \log 10^2 + \log 1{\cdot}353 = 2 + \log 1{\cdot}353$;

$\therefore$ the characteristic is 2.

So also, $\because$ ${\cdot}001353 = 1{\cdot}353 \times 10^{-3}$,

$\therefore$ $\log {\cdot}001353 = -3 + \log 1{\cdot}353$;

$\therefore$ the characteristic is $\bar{3}$.

We can therefore deduce the following rule:—express the given number as the product of a number having one digit only in its integral part and a power of the base 10 (positive or negative); then the characteristic is the index of the power of ten. The mantissa is, of course, positive in all cases, and the new number and given number have the same sequence of digits.

To find the value of x from such an equation as

$$x = \frac{100 \sin 51^\circ \sin 50^\circ}{\sin 16^\circ},$$

we take logarithms and obtain

$$\log x = \log 100 + \log \sin 51^\circ + \log \sin 50^\circ - \log \sin 16^\circ.$$

We may find the values of the sines of 51°, 50°, 16° from the Table of Sines, and then the logarithms of these values from the Table of Logarithms of Numbers. This process is of such frequent occurrence in the *solution of triangles* that it is found very convenient to have the logarithms of the numbers in the Table of Sines (and that of Tangents) calculated once for all and shown in new tables. The two operations indicated above are thus avoided, and one single reference gives the value of such quantities as log sin 51°, log tan 51°.

As the sine of an angle is never greater than unity, the logarithm of the sine will *never be positive*; and the same is true of the cosine. The logarithm of the tangent of an angle will, also, be negative when the angle is less than 45°. In such cases the characteristics will be negative; and to avoid as far as possible the occurrence of negative numbers in the Tables, it is usual to increase each of the logarithms by 10, that is, to multiply the ratios themselves by 10^{10} before taking their logarithms.

Def. The *tabular logarithm* of any trigonometrical function is the common logarithm of that function *increased by* **10**.

Tabular Logarithms are indicated by prefixing L instead of *log* to the functions; and the tables thus obtained are called Tables of Logarithmic Sines, Logarithmic Tangents,..., to distinguish them from the tables of the *numerical values* of the sines, tangents,..., which are called Tables of Natural Sines, Natural Tangents,.......

As in the case of natural cosines and cotangents, logarithmic cosines and cotangents will be given by the Tables of Logarithmic Sines and Tangents, the references being made to the complementary angles. No separate tables are required for logarithmic secants and cosecants also, which may be deduced from the corresponding logarithmic cosines and sines by simple subtraction. For,

$\because$ cosec A × sin A $= 1$, $\therefore$ log cosec A + log sin A $= 0$;
hence, L cosec A + L sin A $= 20$, and L cosec A $= 20 - L$ sin A.
Similarly we have L sec A $= 20 - L$ cos A.

It can also be shown that L cot A + L tan A $= 20$.

The Logarithmic Tables of ratios are used in precisely the same way as the Tables of Logarithms of Numbers or Natural Sines.

Examples. 1. Find $L \sin 55^\circ 26'$ and $L \cos 55^\circ 26'$.

From the Table of Logarithmic Sines

$L \sin 55^\circ 24' = 9{\cdot}9155$;
adding for increase of $2'$, 2,
we get $L \sin 55^\circ 26' = 9{\cdot}9157$.
Also, $L \sin 34^\circ 30' = 9{\cdot}7531$;
adding for increase of $4'$, 7,
we get $L \sin 34^\circ 34' = 9{\cdot}7538$;
hence $L \cos 55^\circ 26' = 9{\cdot}7538$.

It should be noted that the *actual* logarithms of $\sin 55^\circ 26'$ and $\cos 55^\circ 26'$ are $\bar{1}{\cdot}9157$ and $\bar{1}{\cdot}7538$, as the student may verify by using the Tables of Natural Sines and Logarithms of Numbers.

2. Find the angle A from $L \tan A = 8{\cdot}8040$.

From the Table of Logarithmic Tangents

$8{\cdot}7988 = L \tan 3^\circ 36'$;
adding for increase of 52, $3'$
we get $8{\cdot}8040 = L \tan 3^\circ 39'$;
thus angle required $= 3^\circ 39'$.

The increment 52 is not in the Table: it lies between 41 (for $2'$) and 62 (for $3'$); being nearer the latter, we take the increment in angle as $3'$.

3. Find $L \sec 37^\circ 1'$ and the angle whose logarithmic secant is $10{\cdot}41$.

As proved above, $L \sec 37^\circ 1' = 20 - L \cos 37^\circ 1'$
$= 20 - L \sin 52^\circ 59'$;
the Table gives $L \sin 52^\circ 59' = 9{\cdot}9023$;
$\therefore$ $L \sec 37^\circ 1' = 10{\cdot}0977$.

In the second case, if B is the angle required,

$L \cos B = 20 - L \sec B = 9{\cdot}59$.

Now $L \sin 22^\circ 48' = 9{\cdot}5883$;
adding for 17, $6'$,
we have $L \sin 22^\circ 54' = 9{\cdot}5900$:
also $\sin 22^\circ 54' = \cos 67^\circ 6'$.
Hence $10{\cdot}41 = L \sec 67^\circ 6'$, and $B = 67^\circ 6'$.

In this case it is seen that the Table gives $L \sin 22^\circ 54' = 9{\cdot}5901$.

4. Calculate $\sqrt[12]{(1873)}$.

Let x denote the value;
then $\log x = \frac{1}{12} \log 1873 = \frac{1}{12}(3{\cdot}2725) = {\cdot}2727$.
The antilogarithm of ${\cdot}272 = 1871$;
adding for 7 3,
we get $x = 1{\cdot}874$.

5. Obtain the value of β from $7\cos\beta = 12\sin 14^\circ\,14'$.

Taking logarithms and adding 10 to both sides,

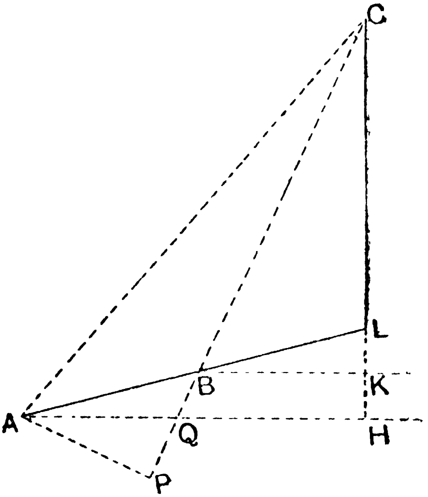
$$\log 7 + L\cos\beta = \log 12 + L\sin 14^\circ\,14'.$$

As $L\sin 14^\circ\,12'$ is $9{\cdot}3897$ and the correction for $2'$ is 10,

we get $L\cos\beta = 9{\cdot}3907 + 1{\cdot}0792 - 8451 = 9{\cdot}6248$.

Thus $90^\circ - \beta = 24^\circ\,54' + 2' = 24^\circ\,56'$, and $\beta = 65^\circ\,4'$.

6. A tower stands on a hillside which has a uniform slope of 15°; from a point A at the foot of the hill the elevation of the top of the tower is found to be 50°, and from a point B, directly up the slope at a distance of 100 ft. from A, it is found to be 66°. Find the height of the top of the tower above the level of the ground.



Let ABL be the slope of the hill; AH, BK the horizontal lines through A and B, and C the top of the tower.

Let CB meet AH in Q, and draw AP perp. to BQ produced. It is given that AB = 100, $\angle$ CAH = 50°, $\angle$ CBK = 66°, $\angle$ BAQ = 15°.

Then $\angle$ ABQ = $\angle$ CQH − $\angle$ BAQ = $\angle$ CBK − $\angle$ BAQ = 51°;
and $\angle$ ACP = $\angle$ CQH − $\angle$ CAH = 16°.

Now $\because \dfrac{AP}{AB} = \sin ABQ$, $\therefore AP = 100\sin 51^\circ$;

and $\dfrac{AC}{AP} = \operatorname{cosec} ACP$, $\therefore AC = AP\operatorname{cosec} 16^\circ$.

Finally, CH, the height required,

$$= AC\sin 50^\circ = AP\operatorname{cosec} 16^\circ \sin 50^\circ$$

$$= 100\sin 51^\circ \operatorname{cosec} 16^\circ \sin 50^\circ = \frac{100\sin 51^\circ \sin 50^\circ}{\sin 16^\circ}.$$

Taking logarithms

$$\log CH = \log 100 + L\sin 51^\circ + L\sin 50^\circ - L\sin 16^\circ - 10$$

$$\begin{array}{rl} = & \left.\begin{array}{l} 2 \\ +\ 9{\cdot}8905 \\ +\ 9{\cdot}8843 \end{array}\right\} - 19{\cdot}4403 \\ = & 21{\cdot}7748 - 19{\cdot}4403 = 2{\cdot}3345. \end{array}$$

The antilogarithm of $2{\cdot}3345$ is $216{\cdot}0$;

thus CH = 216 ft. almost exactly.

Exercises.

119. Find the numbers whose logarithms are $2{\cdot}3457$, $\bar{3}{\cdot}4568$.

120. Obtain $\log_8 (\sin 135^\circ)$, $\log_9 (\tan 60^\circ)$, $\log_{100} ({\cdot}01)^3$.

121. Find the characteristics of
$\log_{10} {\cdot}0527$, $\log_2 {\cdot}527$, $\log_3 5270$, $\log_4 \sqrt{({\cdot}0527)}$.

122. Find the values of $L \sin 80^\circ\, 30'$, $L \cos 80^\circ\, 40'$, $L \tan 80^\circ\, 50'$.

123. Given $L \sin 32^\circ\, 20' = 9{\cdot}72823$, $L \cos 32^\circ\, 20' = 9{\cdot}92683$, find the L tan and L sec of $57^\circ\, 40'$.

124. Find the angles in the following equations :—
$L \cos A = 9{\cdot}8765$, $L \tan B = 11{\cdot}2345$.

125. Find the logarithms of sin $47^\circ 56'$ and cot $56^\circ 47'$, and verify by means of the Tables of Logarithmic Sines and Tangents.

126. Prove that $L \sec A - L \tan A = L \cot A - L \cos A$,
and $L \cot A = 20 - L \tan A = 10 + L \cos A - L \sin A$.

127. Prove the identities—
$2 \log \tan A = \log (\sec A - 1) + \log (\sec A + 1)$,
$L \sin A = 10 + \frac{1}{2} \log (1 - \cos A) + \frac{1}{2} \log (1 + \cos A)$.

128. Prove that $a^{\log b} = b^{\log a}$ and $\log_a b . \log_b a = 1$.

129. Given $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}47712$, find the number of digits in 2^{64} and 15^{15}.

130. Given $\log 2 = {\cdot}30103$, $\log 7 = {\cdot}84510$, find the position of the first significant figure in $({\cdot}05)^{15}$ and $\left(\frac{4}{343}\right)^{24}$.

131. Given $\log 45 = p$, $\log 54 = q$, find log 2 and log 3 in terms of p and q.

132. Given $\log 2 = a$, $\log 3 = b$, $\log 5 = c$, $\log 7 = d$ to any base find to the same base $\log ({\cdot}018)$, $\log (15{\cdot}12)^2$, $\log (1764)^{\frac{1}{5}}$.

133. Given $\log_{10} 3 = {\cdot}47712$, find x from the equation

$$(3\tfrac{1}{8})^{x+2} = 9^{x-1}.$$

134. By means of the Tables solve the equation

$$4^x + 4^{-x} = 40{\cdot}025.$$

135. Employ the Tables to obtain the values of

$$23{\cdot}62 \tan 27^\circ 11' \operatorname{cosec} 30^\circ 40' \quad \text{and} \quad \frac{2362 \sin 65^\circ 8'}{\sin 40^\circ 30'}.$$

136. Calculate $\dfrac{\cos^2 \lambda}{289}$ when $\lambda = 21^\circ 30'$; also $2\pi \div \sqrt{g}$, where $\pi = 3{\cdot}142$ and $g = 32{\cdot}09$.

137. Given $a = 5{\cdot}1$, $b = 3{\cdot}4$, $c = 7{\cdot}08$, and $2s = a + b + c$, calculate $\sqrt{\dfrac{(s-b)(s-c)}{bc}}$ and $\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}$.

Solve the following problems (138–142), employing as far as possible the Tables of Logarithms and Logarithmic functions:—

138. Find the side of a regular polygon of 24 sides described about a circle of radius 13 mm.

139. The elevation of a spire whose height is 250 ft. is 34° 17′: find the distance of its top from the observer.

140. A river is 205 feet wide, and the angles of depression of its banks, observed from a point in a vertical plane to which the banks are both perpendicular, are 20° 17′ and 17° 20′: find the height above the ground of the point of observation.

141. A circular tower, whose diameter is known to be 72 ft., subtends an angle of 10° at the eye of an observer on level ground, and the elevation of its top is found to be 36° 25′: find the height of the tower.

142. A spherical balloon, 5 metres in radius, subtends at the eye of an observer an angle of 1° 46′, while the elevation of its centre is found to be 25° 40′: find the vertical height of the balloon.

CHAPTER XIV.

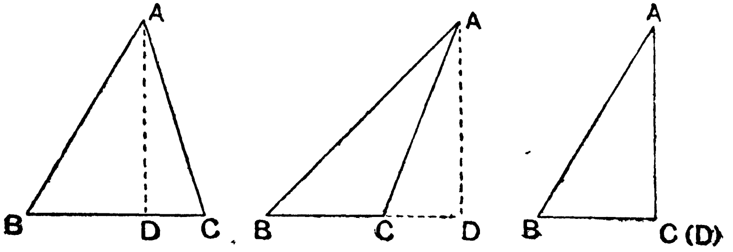
TRIANGLE FORMULÆ.

§32. FUNDAMENTAL FORMULÆ RELATING TO TRIANGLES.

In the two preceding sections we have given methods for the solution of right-angled triangles, and have shown how oblique-angled triangles may be solved by being resolved into the sum or difference of right-angled triangles. In this and the following section we shall give the algebraical formulæ required in the numerical solution of triangles of any sort. As before, in the geometrical triangle ABC the measures of the angles will be taken to be A, B, C, and the lengths of the respectively opposite sides to be a, b, c.

I. *In any triangle the sides are proportional to the sines of the opposite angles*; that is,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$



Draw AD perp. to BC or BC produced. Suppose B to be acute; then, in all cases,

$$\frac{DA}{AB}=\sin B, \quad \therefore DA=AB\sin B=c\sin B.$$

In the first figure, where C is acute,

$$\text{DA} = \text{AC} \sin \text{C} = b \sin \text{C};$$

in the second, where C is obtuse,

$$\text{DA} = \text{AC} \sin \text{ACD} = \text{AC} \sin (180^\circ - \text{C}) = b \sin \text{C}, \quad §26;$$

in the third, where C is right, DA is the same as CA,

$$\therefore \quad \text{DA} = b = b \sin 90^\circ = b \sin \text{C}, \text{ as } \sin 90^\circ = 1.$$

Thus in all cases $c \sin \text{B} = b \sin \text{C}$,

$$\text{or } \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}}.$$

Similarly, by drawing the perp. from B on CA or CA produced,

$$\text{we can prove } \frac{c}{\sin \text{C}} = \frac{a}{\sin \text{A}}.$$

$$\text{Hence } \frac{a}{\sin \text{A}} = \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}}.$$

Cor. Putting each fraction $= d$, we have

$$a = d \sin \text{A}, \quad b = d \sin \text{B}, \quad c = d \sin \text{C}.$$

Thus each side is expressed in terms of the sine of the opposite angle and one length, d, which will be proved later on to be the diameter of the circumcircle of the triangle.

The theorem proved above is often called the *Rule of Sines*: as $a \sin \text{B} = b \sin \text{A}$, $b \sin \text{C} = c \sin \text{B}$, $c \sin \text{A} = a \sin \text{C}$, it may also be called the *double sine formula*. Although we get three equations, it should be noticed that there are only *two independent* relations.

II. *In any triangle the square of a side is less than the sum of the squares of the other two sides by twice the rectangle contained by these sides and the cosine of the included angle*; that is,

$$a^2 = b^2 + c^2 - 2bc \cos \text{A},$$
$$b^2 = c^2 + a^2 - 2ca \cos \text{B},$$
$$c^2 = a^2 + b^2 - 2ab \cos \text{C}.$$

In the first figure above, as C is acute

$$\text{AB}^2 = \text{BC}^2 + \text{CA}^2 - 2\ \text{BC.CD};$$

but $\frac{\text{CD}}{\text{CA}} = \cos \text{C}$, so that $\text{CD} = \text{CA} \cos \text{C}$;

$$\therefore \quad \text{AB}^2 = \text{BC}^2 + \text{CA}^2 - 2\ \text{BC.CA} \cos \text{C};$$

that is, $c^2 = a^2 + b^2 - 2ab \cos \text{C}$.

In the second figure, where C is obtuse,

$$\text{AB}^2 = \text{BC}^2 + \text{CA}^2 + 2\ \text{BC.CD};$$

but $\text{CD} = \text{CA} \cos \text{ACD} = \text{CA} \cos (180^\circ - \text{C}) = -\text{CA} \cos \text{C}$, § 26,

$$\therefore \text{AB}^2 = \text{BC}^2 + \text{CA}^2 - 2\ \text{BC. CA} \cos \text{C}$$

that is, $c^2 = a^2 + b^2 - 2ab \cos \text{C}$.

In the third figure, where C is right,

$$AB^2 = BC^2 + CA^2 = BC^2 + CA^2 - 2\ BC.CA \cos C,$$

as $\cos 90^\circ = 0$. Hence also $c^2 = a^2 + b^2 - 2ab \cos C$.

Thus the formula is true in all cases. The other two formulæ may be similarly demonstrated.

Cor. By transposing we get $2ab \cos C = a^2 + b^2 - c^2$,

$$\text{hence } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

$$\text{So also} \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

These formulæ enable us to express the cosine of any angle of a triangle in terms of the sides; and the theorem given above is often called the *Cosine Rule*. It is seen to be the general algebraical expression of the theorems [29],[30], [31] in § 1 (*Geometry*).

Examples. 1. Of a triangle ABC given $A = 60^\circ$, $B = 45^\circ$, $b = 6$; find the other sides.

Here $C = 75^\circ$; and as $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$,

$$\therefore c = \frac{b \sin C}{\sin B} = \frac{6 \sin 75^\circ}{\sin 45^\circ} = \frac{6 \cos 15^\circ}{\sin 45^\circ} = \frac{6(\sqrt{6} + \sqrt{2})}{2\sqrt{2}}$$

$$= 3(\sqrt{3} + 1); \text{ see § 27, Exam. 2.}$$

$$\text{And } a = \frac{b \sin A}{\sin B} = \frac{6 \sin 60^\circ}{\sin 45^\circ} = 3\sqrt{6}.$$

2. Given $a = 5$, $b = 8$, $B = 60^\circ$; find the other angles.

$$\text{We have } \frac{\sin A}{a} = \frac{\sin B}{b}; \quad \therefore \sin A = \frac{a \sin B}{b} = \frac{5\sqrt{3}}{16};$$

reducing this to a decimal, ·5413, we find from the Tables $A = 32^\circ 47'$.

Hence $C = 87^\circ 13'$.

The angle A may also be taken to be $147^\circ 13'$, as $\sin(180^\circ - A) = \sin A$. This value is, however, inapplicable here, as $A + B = 207^\circ 13'$ would be impossible in a real triangle. But in other similar cases both values may be applicable, and the solution may be really ambiguous.

3. Find the angles of a triangle whose sides are 3, 4, 6. Denoting these respectively by a, b, c, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{43}{48} = ·8958; \quad \therefore A = 26^\circ 23' \text{ from the Tables.}$$

So $\cos B = \dfrac{29}{36} = ·8056$, and $B = 36^\circ 20'$. Hence $C = 117^\circ 17'$.

If we find C first, we get $\cos C = -\frac{11}{24} = -·4583$; whence $C = 180^\circ - 62^\circ 43' = 117^\circ 17'$, as $\cos(180^\circ - C) = -\cos C$.

There is no ambiguity when an angle of a triangle is determined from its cosine; if the cosine is positive the angle is acute, and if negative it is obtuse. The same remark applies to the tangent. The case is, however, different with the sine, which is positive for both acute and obtuse angles; and further investigation is necessary to find out if there are really two solutions or not.

The following algebraical result is of use in the next proposition.

The expression $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
may be written as $4b^2c^2-(a^4+b^4+c^4+2b^2c^2-2c^2a^2-2a^2b^2)$,
that is, as $(2bc)^2-(b^2+c^2-a^2)^2$;
hence it is the product of

$$2bc+b^2+c^2-a^2,\quad 2bc-b^2-c^2+a^2.$$

But these factors are $(b+c)^2-a^2$, $a^2-(b-c)^2$, respectively; so that the original expression is the product of

$$a+b+c,\ -a+b+c,\ a-b+c,\ a+b-c.$$

Now let $a+b+c=2s$; $\therefore 2(s-a)=-a+b+c$, etc.

Hence
$$\begin{aligned}&2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4\\ =\ &(a+b+c)(-a+b+c)(a-b+c)(a+b-c)\\ =\ &16\,s\,(s-a)(s-b)(s-c).\end{aligned}$$

III. *To express the sine of an angle of a triangle in terms of the sides.*

Since $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$,

$$\therefore\quad \sin^2 A=1-\cos^2 A=\frac{4b^2c^2-(b^2+c^2-a^2)}{4b^2c^2}.$$

The numerator $=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
$=16\,s\,(s-a)(s-b)(s-c)$, as shown above,
s being $\frac{1}{2}(a+b+c)$ or the *semi-perimeter* of the triangle.

$$\text{Hence}\quad \sin A=\frac{1}{2bc}\sqrt{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)}$$

$$\text{or}\quad \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

the positive sign being taken as sin A is positive whether A be acute or obtuse.

$$\text{Similarly}\ \sin B=\frac{2}{ca}\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

$$\text{and}\quad \sin C=\frac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Hence also we conclude that $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

IV. *To find expressions for the area of a triangle.*

As proved in I above, the perp. AD is equal to $b \sin C$ in all cases.

Now the area is $\frac{1}{2}$BC.AD [§ 1, 24]; hence, denoting it by Δ, we have

$$\Delta = \tfrac{1}{2} ab \sin C,$$
$$= \tfrac{1}{2} bc \sin A \text{ or } \tfrac{1}{2} ca \sin B \text{ by I above.}$$

Also, substituting the value of sin A from III, we get

$$\Delta = \tfrac{1}{2} bc . \frac{2}{bc} \sqrt{\{s(s-a)(s-b)(s-c)\}}$$
$$= \sqrt{\{s(s-a)(s-b)(s-c)\}}. \text{ (See } \textit{Geometry}, \text{ Exer. 8.)}$$

The area can be found in terms of one side alone: for

$$b = \frac{a \sin B}{\sin A} \quad \text{from I; hence}$$

$$\Delta = \tfrac{1}{2} a \frac{a \sin B}{\sin A} \sin C = \tfrac{1}{2} a^2 \frac{\sin B \sin C}{\sin A},$$

$$= \tfrac{1}{2} b^2 \frac{\sin C \sin A}{\sin B} \text{ or } \tfrac{1}{2} c^2 \frac{\sin A \sin B}{\sin C} \quad \text{similarly.}$$

V. In any triangle

$$a = b \cos C + c \cos B,$$
$$b = c \cos A + a \cos C,$$
$$c = a \cos \quad + b \cos A.$$

In the first figure of I, we have

$$CD = b \cos C,\ BD = c \cos B;$$
$$\therefore b \cos C + c \cos B = BD + CD = BC = a.$$

In the second figure, $CD = -b \cos C$ as proved; $BD = c \cos B$,

$$\therefore b \cos C + c \cos B = BD - CD = BC = a.$$

When C is right, $BC = c \cos B$; $b \cos C = 0$ for $\cos 90° = 0$;

$$\therefore b \cos C + c \cos B = BC = a.$$

Thus the result is true in all cases; and the other two results can be similarly proved.

In I, II and V we have separately derived eight relations between the sides and angles of a triangle; and $A + B + C = 180°$ is a ninth relation. These relations are not all independent; for a triangle is known if three of its parts (including a side) are given, and to determine three parts three equations ought to suffice. Hence from three of these nine relations it is possible to derive the remaining six; for instance, we see in III how the *sine rule* can be obtained from the *cosine formulæ*. The process is in general difficult; some simple examples will be found in the exercises below.

Examples (*Contd.*). 4. Prove that the angle C of a triangle ABC is right, acute, obtuse according as c^2 is equal to, less than, greater than a^2+b^2.

This well-known geometrical result may be thus proved.—

If $c^2=a^2+b^2$, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}=0$; $\therefore$ $C=90°$.

If $c^2<a^2+b^2$, cos C is *positive*, and C is *acute*;

if $c^2>a^2+b^2$, cos C is *negative*, and C is *obtuse*.

5. Show that $\tan B : \tan C=a^2+b^2-c^2 : a^2+c^2-b^2$.

Here $\dfrac{\tan B}{\tan C}=\dfrac{\sin B}{\cos B}\div\dfrac{\sin C}{\cos C}=\dfrac{\sin B}{\sin C}\cdot\dfrac{\cos C}{\cos B}$

$$=\frac{b}{c}\cdot\frac{(a^2+b^2-c^2).\,2ca}{2ab.\,(c^2+a^2-b^2)}=\frac{a^2+b^2-c^2}{a^2+c^2-b^2}.$$

6. Prove the identities $b(\tan B+\tan C)=a\tan B\sec C$, and $c(\tan B+\tan C)=a\tan C\sec B$.

$$\because a\tan B\sec C-b\tan B=\tan B\left(\frac{a}{\cos C}-b\right)$$

$$=\tan B\frac{a-b\cos C}{\cos C}$$

$$=\frac{\tan B.\,c\cos B}{\cos C},\text{ by V above,}$$

$$=\frac{c\sin B}{\cos C}=\frac{b\sin C}{\cos C},\text{ by I above,}$$

$$=b\tan C;$$

$\therefore a\tan B\sec C=b(\tan B+\tan C)$.

This proves the first; and the second identity may be similarly proved.

7. With the ordinary notation for a triangle, prove that $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$.

$\because \cos A=\dfrac{b^2+c^2-a^2}{2bc}$, $\sin A=\dfrac{2\Delta}{bc}$, by II and IV;

$\therefore \cot A=\dfrac{b^2+c^2-a^2}{4\Delta}$, with analogous values for cot B and cot C.

$\therefore$ the given left side

$$=\frac{1}{4\Delta}\{(b^2-c^2)(b^2+c^2-a^2)+(c^2-a^2)(c^2+a^2-b^2)$$
$$+(a^2-b^2)(a^2+b^2-c^2)\}$$
$$=\frac{1}{4\Delta}\{b^4-c^4-a^2(b^2-c^2)+c^4-a^4-b^2(c^2-a^2)+a^4-b^4$$
$$-c^2(a^2-b^2)\}$$
$$=0.$$

8. In any triangle prove that

$$\frac{b-c}{a^2}\sin A+\frac{c-a}{b^2}\sin B+\frac{a-b}{c^2}\sin C=-\frac{2\Delta(b-c)(c-a)(a-b)}{a^2b^2c^2}.$$

Substituting $\frac{2\Delta}{bc}$ for $\sin A$, and similar values for $\sin B$, $\sin C$, the left side gives

$$\frac{2\Delta(b-c)}{a^2bc}+\frac{2\Delta(c-a)}{b^2ca}+\frac{2\Delta(a-b)}{c^2ab}$$

$$=\frac{2\Delta}{a^2b^2c^2}\{bc(b-c)+ca(c-a)+ab(a-b)\}$$

$$=-\frac{2\Delta}{a^2b^2c^2}(b-c)(c-a)(a-b);$$

and this is the right side. (See Hall and Knight, *Elem. Algebra*, § 223.)

Exercises.

143. The vertical angle of an isosceles triangle is 120°, and the base is 10: find the sides of the triangle.

144. The base of an isosceles triangle is 20, and its area is $\frac{100}{3}\sqrt{3}$: find the angles of the triangle.

145. In a triangle given $c=10$, $b=9$, $B=60°$; find C and A.

146. Given $c=10$, $a=9$, $B=45°$; find b.

147. The sides of a triangle are n^2+n+1, n^2-1, $2n+1$, where $n>1$: prove that the greatest angle is 120°.

148. Two adjacent sides of a parallelogram are 2 and $\sqrt{3}$ in length, and the included angle is 30°: find both the diagonals.

149. Find the resultant of two forces (P,Q) acting on a particle at the angle (A), as specified below:—

(i) $P=4$ lbs., $Q=5$ lbs., $A=60°$;

(ii) $P=3\sqrt{2}$ *kgs.*, $Q=\frac{3}{2}$ *kgs.*, $A=135°$;

(iii) $P=18$ *grs.*, $Q=23$ *grs.*, $A=75°$.

[When two forces acting at a point are represented by straight lines drawn through the point, it is known that their resultant is represented by the line joining the point to the opposite corner of the parallelogram formed on the two straight lines as adjacent sides.]

150. With the usual notation for a triangle prove that

$$\frac{b^{\frac{3}{2}}+c^{\frac{3}{2}}}{\sqrt{(bc)}}\cos A+\frac{c^{\frac{3}{2}}+a^{\frac{3}{2}}}{\sqrt{(ca)}}\cos B+\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{\sqrt{(ab)}}\cos C=a+b+c.$$

151. Find the area of the triangle ABC, given that—
(i) $a=1{\cdot}3,\ b=1{\cdot}4,\ c=1{\cdot}5$; (ii) $a=4,\ b=23{\cdot}1,\ c=25{\cdot}7$;
(iii) $a=60,\ b=50,\ C=60^\circ 50'$; (iv) $c=35,\ A=30^\circ,\ B=45^\circ$.

152. A triangle of area 756 sq. yds. has two sides of lengths 189 ft., 222 ft: find the length of the third side.

153. The sides of a triangle are 35, 78, 97: find the perpendicular on the mean side from the opposite vertex.

154. Points P, Q, R are respectively taken in the sides BC, CA, AB of a triangle, such that

$$BP : PC = CQ : QA = AR : RB = 2 : 5;$$

prove that

$$\Delta PQR = \frac{19}{49} \Delta ABC,$$

and

$$PQ^2 + QR^2 + RP^2 = \frac{19}{49}(a^2+b^2+c^2).$$

155. If, in a triangle, $b+c : c+a : a+b = 10 : 11 : 14$, find the ratios sin A : sin B : sin C ; and show that the mean angle of the triangle is 60°.

156. If a triangle ABC is right-angled at C, prove that

$$\cos^2 \frac{A}{2} = \frac{b+c}{2c},$$

and

$$4\Delta^2 = abc^2 \cos A \cos B.$$

In any triangle ABC prove the following relations:— (157–162)

157. $\tan B = b \sin C \div (a - b \cos C)$.

158. $c(a \cos B - b \cos A) = a^2 - b^2$.

159. $b \sin^2 C = c(\cos A + \cos B \cos C)$.

160. $ab \sin^2 C = c(a \cos B \cos C + b \cos C \cos A + c \cos A \cos B)$.

161. $(b \sec B - c \sec C) \cos A = c \sec B - b \sec C$.

162. $$\frac{a - b\cos C}{ab^2} + \frac{b - c\cos A}{bc^2} + \frac{c - a\cos B}{ca^2} = \frac{a^4+b^4+c^4}{2a^2b^2c^2}.$$

163. Prove that

$$\frac{1+\cos A}{a} + \frac{1+\cos B}{b} + \frac{1+\cos C}{c} = \frac{(a+b+c)^2}{2abc},$$

and obtain a similar expression for

$$\frac{1-\cos A}{a} + \frac{1-\cos B}{b} + \frac{1-\cos C}{c}.$$

*164. Prove that

$$\frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s} = \frac{2b}{\sin B},$$

and $$4\Delta s = abc\{\sin A + \sin B + \sin A \cos B + \sin B \cos A\}.$$

165. If p, q, r are the lengths of the perpendiculars from A, B, C to BC, CA, AB respectively, prove that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{s}{\Delta},$$

and $2p\cos A+2q\cos B+2r\cos C=a\sin A+b\sin B+c\sin C$.

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166. From the equalities—

$$a=b\cos C+c\cos B,$$
$$b=c\cos A+a\cos C,$$
$$c=a\cos B+b\cos A,$$

deduce the *Cosine Rule.*

167. Show how to obtain the *Sine Rule* from the relations given above; and prove also that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

*168. Prove that $4\Delta(\cot A+\cot B+\cot C)=a^2+b^2+c^2$,
and $2\Delta=bc(\sin B\cos C+\sin C\cos B)$.

From the latter identity deduce the result—

$$\sin(B+C)=\sin B\cos C+\sin C\cos B.$$

169. If P divides BC so that BP : PC $=m:n$, prove that

$$(m+n)^2\,AP^2=m^2b^2+n^2c^2+2mnbc\cos A.$$

170. The sides of a parallelogram are a, b in length, and include an angle θ: prove that the product of the diagonals is

$$\sqrt{\{(a^2-b^2)^2+4a^2b^2\sin^2\theta\}}.$$

171. In any triangle prove that

$$(b^2+ca)\cos A+(c^2+ab)\cos B+(a^2+bc)\cos C$$
$$=(c^2+ab)\cos A+(a^2+bc)\cos B+(b^2+ca)\cos C$$
$$=bc+ca+ab, \text{ and that}$$
$$abc(3+\cos A+\cos B+\cos C)=s(2ab+2bc+2ca-a^2-b^2-c^2).$$

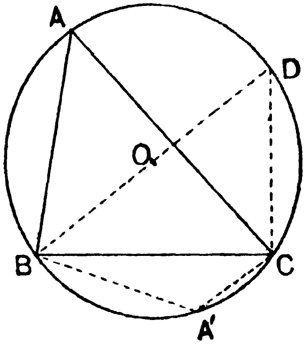
172. If in a triangle a^2, b^2, c^2 are in A.P., then tan A, tan B, tan C are in H. P.

173. The length of the median AD of a triangle ABC is l; prove that $4l^2=b^2+c^2+2bc\cos A$.

Hence show that if m, n are the lengths of the other two medians, $4(l^2+m^2+n^2)=3(a^2+b^2+c^2)$. (See Exer. 12, *Geometry.*)

§ 33. CIRCLES CONNECTED WITH TRIANGLES: HALF-ANGLE FORMULÆ.

I. *To find expressions for the radius (R) of the circumcircle of a given triangle.*



Let O be the circumcentre of the Δ ABC; draw BOD the diameter through B. Then ∠ BCD = 90°, in a semicircle; and ∠ BDC = ∠ BAC or A, in the same segment.

Hence $\sin A = \sin BDC = \frac{BC}{BD} = \frac{a}{2R}$;

$$\therefore\ R = \frac{a}{2\sin A}.$$

Similarly we shall have $R = \frac{b}{2\sin B}\quad\frac{c}{2\sin C}$.

If the angle opposite to BC is obtuse, as in the Δ A′BC, we have ∠ BDC = 180° − BA′C;

and ∵ $\sin BDC = \sin(180° - BA'C) = \sin BA'C$,

∴ we shall obtain the same result.

Again, ∵ $\sin A = \frac{2\Delta}{bc}$, as proved in the preceding section,

we get $R = \frac{a}{4\Delta \div bc} = \frac{abc}{4\Delta}$. (See *Geometry*, XI., *Cor.*)

This gives R in terms of the sides in the form

$$\frac{abc}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}};$$

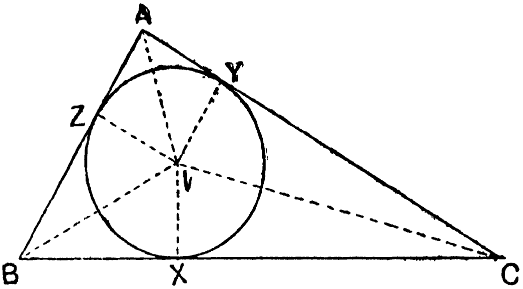
we also see that each of the fractions $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, $\frac{c}{\sin C}$ represents the diameter of the circumcircle.

Cor. As BC is a chord of the circle of radius R subtending an angle A at the circumference, we see that the length of the chord of a circle which subtends an angle α at the circumference is

$$2R \sin \alpha \text{ or } d \sin \alpha,$$

where d is the diameter.

II. *To find expressions for the radius* (r) *of the incircle of a triangle.*



Let I be the incentre of the ΔABC; X, Y, Z its points of contact in BC, CA, AB. Join AI, BI, CI.

Then area BIC $= \frac{1}{2}$ BC. IX, as the angles at X are right,

$= \frac{1}{2} ar.$

So also area CIA $= \frac{1}{2} br$, area AIB $= \frac{1}{2} cr$.

$\therefore$ adding up, area ABC or $\Delta = \frac{1}{2}(a+b+c)\,r = sr$;

hence $$r = \frac{\Delta}{s}.$$

Again, it is known that $\text{AZ} = \text{AY} = \frac{1}{2}(\text{AZ} + \text{AY})$;

so also, $\text{BX} = \frac{1}{2}(\text{BX} + \text{BZ})$,

$\text{CX} = \frac{1}{2}(\text{CX} + \text{CY})$.

$\therefore$ adding up, $\text{AZ} + (\text{BX} + \text{CX}) = \frac{1}{2}(\text{AZ} + \text{AY} + \text{BX} + \text{BZ} + \text{CX} + \text{CY})$,

or $\text{AZ} + a = \frac{1}{2}(a+b+c)$, so that $\text{AZ} = s - a$.

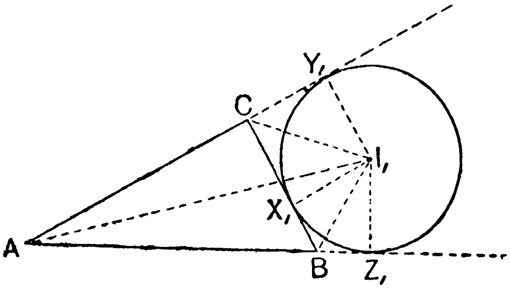
Now Δs AZI, AYI are congruent, so that $\angle \text{ZAI} = \frac{1}{2}$ A.

Hence, $\tan\frac{\text{A}}{2} = \frac{\text{IZ}}{\text{AZ}} = \frac{r}{s-a}$; $\therefore\ r = (s-a)\tan\frac{\text{A}}{2}$.

Similarly also $r = (s-b)\tan\frac{\text{B}}{2}$ or $(s-c)\tan\frac{\text{C}}{2}$.

(See *Geometry*, XVIII and Exer. 160.)

III. *To find expressions for the radius* (r_1) *of the escribed circle of a triangle* ABC, *touching* BC *and the other two sides produced.*



Let I_1 be the centre of this circle; X_1, Y_1, Z_1 its points of contact in BC, CA, AB. Join AI_1, BI_1, CI_1.

Then area $AI_1B = \frac{1}{2} AB.I_1Z_1 = \frac{1}{2} c\, r_1$;

so also area $BI_1C = \frac{1}{2} a\, r_1$, area $AI_1C = \frac{1}{2} b\, r_1$.

Hence $\frac{1}{2} cr_1 + \frac{1}{2} br_1 - \frac{1}{2} ar_1 =$ area $AI_1B +$ area $AI_1C -$ area BI_1C

$=$ area ABC $= \Delta$;

$\therefore\ r_1\ (s-a) = \Delta$, and $r_1 = \dfrac{\Delta}{s-a}$.

Similarly, the radii r_2, r_3 of the excircles touching CA, AB respectively and the remaining sides produced, can be shown to be $\dfrac{\Delta}{s-b}$, $\dfrac{\Delta}{s-c}$.

Again, $\because\ 2s = AB + BC + CA$,

and $BC = BX_1 + CX_1 = BZ_1 + CY_1$,

$\therefore\ 2s = (AB + BZ_1) + (CA + CY_1)$

$= AZ_1 + AY_1 = 2AZ_1$;

$\therefore AZ_1 = s$.

But AI_1 bisects the angle BAC,

$\therefore\ \tan\frac{A}{2} = \dfrac{Z_1I_1}{AZ_1} = \dfrac{r_1}{s}$; hence $r_1 = s\tan\frac{A}{2}$.

Similarly also, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$.

(See *Geometry*, Notes 1 and 4 after XVIII.)

IV. *To express the tangents of the half-angles of a triangle in terms of the sides.*

We have proved in II above that

$$r = \frac{\Delta}{s},\ r = (s-a)\tan\frac{A}{2};$$

$$\therefore\ \tan\frac{A}{2} = \frac{\Delta}{s(s-a)}.$$

But $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$;

$$\therefore\ \tan\frac{A}{2} = \frac{\sqrt{\{s(s-a)(s-b)(s-c)\}}}{s(s-a)} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly we shall have

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

These formulæ may also be obtained from the equalities

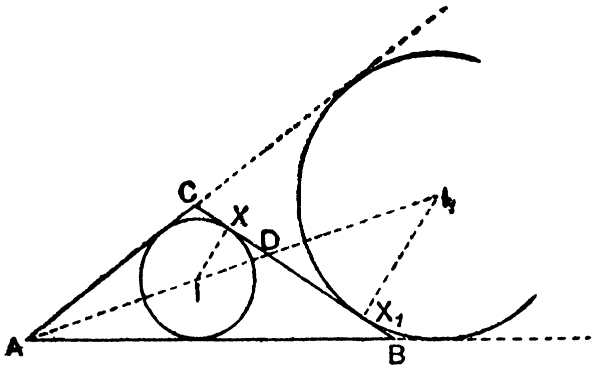
$$r_1 = \frac{\Delta}{s-a},\ r_1 = s\tan\frac{A}{2},\ \text{etc.}$$

They can be employed in determining the angles of a triangle when all its sides are given; see Exam. 4 solved below.

The student should hence find the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\sin\frac{B}{2}$, etc.

V. *To express the tangent of half the difference of two angles of a triangle in terms of the opposite sides and the angle contained by them.*

Let the incircle and the excircle opposite to A touch BC in X, X_1 respectively. If I, I_1 are their centres, AII_1 is a str. line, as it bisects the ∠ BAC; let it meet BC in D.



As in II and III above, we can prove that

$$BX = s - b,\ BX_1 = s - c;$$

$$\therefore\ XX_1 = BX - BX_1 = c - b.$$

Now $\angle I_1DX_1 = \angle IDX = \angle ABD + \angle BAD = B + \frac{A}{2}$

$$= \frac{B}{2} + \frac{C}{2} + \frac{A}{2} - \left(\frac{C}{2} - \frac{B}{2}\right) = 90^\circ - \frac{C-B}{2};$$

$$\therefore\ \angle X_1I_1D = \angle XID = \frac{C-B}{2}.$$

And $\frac{XD}{XI} = \tan XID = \tan\frac{C-B}{2},\quad \frac{X_1D}{X_1I_1} = \tan\frac{C-B}{2},$

$$\therefore\ XX_1 = XD + X_1D = (XI + X_1I_1)\tan\frac{C-B}{2}.$$

Hence $c - b = (r + r_1)\tan\frac{C-B}{2}$ ....(1)

But $r = (s-a)\tan\frac{A}{2},\quad r_1 = s\tan\frac{A}{2}$, as proved above;

$$\therefore\quad r + r_1 = (2s - a)\tan\frac{A}{2} = (b+c)\tan\frac{A}{2}.$$

Hence finally, from (1),

$$c - b = (c+b)\tan\frac{A}{2}\tan\frac{C-B}{2},$$

so that $\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}.$

Similarly, with the figure of the triangle taken above, we can prove that

$$\tan\frac{B-A}{2} = \frac{b-a}{b+a}\cot\frac{C}{2},\quad \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}.$$

These relations can be employed in solving a triangle when two sides and the included angle are given; see Exam. 5 solved below.

Examples. 1. Find the distances of the circumcentre of a triangle, whose sides are 19, 20, 37 in length, from the vertices and the sides of the triangle.

Let the given lengths be those of a, b, c respectively. The sum of the sides is 76; hence $s = 38$:

$\therefore s - a = 19,\ s - b = 18,\ s - c = 1.$

Thus $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$

$= \sqrt{\{38.19.18.1\}} = \sqrt{\{19^2.6^2\}} = 114;$

and $R = \frac{abc}{4\Delta} = \frac{19.20.37}{456} = 30\frac{5}{6}:$

this is the distance of the circumcentre from each vertex.

To find the distance from the side a, we observe that the perp. from the circumcentre bisects the side; hence, the sq. of the distance

$$=\left(\frac{185}{6}\right)^2-\left(\frac{19}{2}\right)^2=\frac{185^2-57^2}{6^2}$$

$$=\frac{242.128}{6^2}=\frac{11^2\,8^2}{3^2};$$

$\therefore$ distance from $a=\dfrac{88}{3}=29\dfrac{1}{3}$.

The distances from b and c will be found to be $29\frac{1}{6}$, $24\frac{2}{3}$.

2. In a triangle, $a=104$, $b=153$, $c=185$: find the radius of the incircle, and the ratios of the segments of the sides made by the points of contact.

$\because\ a+b+c=442, \therefore s=221;$
and $s-a=117,\ s-b=68,\ s-c=36.$

Hence $\Delta=\sqrt{(221.117.68.36)}=\sqrt{(36^2.17^2.13^2)}=36.17.13;$

and $r=\dfrac{\Delta}{s}=\dfrac{17.13.36}{17.13}=36.$

If X is the point of contact in BC, $\text{BX}=s-b=68$ and $\text{XC}=s-c=36$;
$\therefore$ BX : XC $=68:36=17:9$
So also CY : YA and AZ : ZB can be found.

3. In any triangle prove, with the usual notation, that

$$2R^2\sin\text{A}\sin\text{B}\sin\text{C}=\Delta,$$

and

$$r_1+r_2+r_3-r=4R.$$

$\because\ R=\dfrac{a}{2\sin\text{A}}=\dfrac{b}{2\sin\text{B}},\ \therefore\ R^2=\dfrac{ab}{4\sin\text{A}\sin\text{B}},$

so that $2R^2\sin\text{A}\sin\text{B}=\frac{1}{2}ab.$

Hence $2R^2\sin\text{A}\sin\text{B}\sin\text{C}=\frac{1}{2}ab\sin\text{C}=\Delta$, as already proved.
Secondly, we have to prove that

$$\frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}-\frac{\Delta}{s}=\frac{abc}{\Delta},$$

that is,

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{\Delta^2}$$

$$=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

This algebraical result can be obtained without difficulty : see Hall and Knight, *Algebra*, § 223. (Compare Exer. 164 above.)

4. The sides of a triangle are 4, 7, 9: find all the angles.

Here $s=10,\ s-a=6,\ s-b=3,\ s-c=1.$

Hence $\tan^2\dfrac{\text{A}}{2}=\dfrac{(s-b)\,(s-c)}{s\,(s-a)}=\dfrac{1}{20};$

so that $2\log\tan\dfrac{\text{A}}{2}=-\log 20=-1{\cdot}3010.$

Dividing by 2 and adding 10, we have

$$\text{L}\tan\frac{\text{A}}{2}=9{\cdot}3495;\ \therefore\ \frac{\text{A}}{2}=12^\circ 36' \text{ from the Tables.}$$

Hence $\text{A}=25^\circ\,12'$.

Similarly $\tan^2\frac{B}{2}=\frac{(s-c)(s-a)}{s\,(s-b)}=\frac{1}{5}$;

this gives $\frac{B}{2}=24°6'$, and $B=48°12'$.

Finally, $C=180°-A-B=106°36'$.

The angle C should also be obtained from

$$\tan^2\frac{C}{2}=\frac{(s-a)(s-b)}{s\,(s-c)},$$

and the result compared with the value found: this serves as a useful check on the calculation.

5. In a triangle, $A=60°$ and $\frac{c}{b}=1\cdot\dot{2}$: find the other angles.

$\because\ \frac{c}{b}=\frac{11}{9}$, $\therefore\ \frac{c-b}{c+b}=\frac{1}{10}$.

Thus $\tan\frac{C-B}{2}=\frac{1}{10}\cot 30°=\frac{\sqrt{3}}{10}$, by V above.

$\therefore\ L\tan\frac{C-B}{2}=10+\frac{1}{2}\log 3-1=9\cdot2386.$

Hence, from the Tables, $\frac{C-B}{2}=9°50'$;

also $\frac{C+B}{2}=90°-\frac{A}{2}=60°$. By adding and subtracting, we get at once $C=69°50'$, $B=50°10'$.

6. To find the cosines of the half-angles of a triangle in terms of the sides.

We have from IV above, $\tan^2\frac{A}{2}=\frac{(s-b)(s-c)}{s\,(s-a)}$;

$$\therefore\ \sec^2\frac{A}{2}=1+\tan^2\frac{A}{2}=\frac{s\,(s-a)+(s-b)(s-c)}{s\,(s-a)}$$

$$=\frac{2s^2-s(a+b+c)+bc}{s\,(s-a)}$$

$$=\frac{bc}{s(s-a)}.$$

Hence $\cos^2\frac{A}{2}=\frac{s\,(s-a)}{bc}$, and $\cos\frac{A}{2}=\sqrt{\frac{s\,(s-a)}{bc}}$, the positive sign being taken as $\frac{1}{2}$ A is acute.

So also we shall have $\cos\frac{B}{2}=\sqrt{\frac{s\,(s-b)}{ca}}$, $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$.

Exercises.

174. Calculate the radii of the circumcircles and incircles of triangles whose sides are—

(i) 49, 1200, 1201 ; (ii) 50, 369, 401.

175. ABC is a triangle and AX any straight line meeting BC : prove that the circumradii of the triangles ABX, ACX are in a fixed ratio for all positions of AX.

176. O is the circumcentre and P the orthocentre of the triangle ABC : prove that the circles described about the triangles PBC, PCA, PAB are equal, and find the radii of the circles about OBC, OCA, OAB.

177. The area of a triangle is 840, and the radii of two of its excircles are 10·5, 40 : find its sides.

178. The semi-perimeter of a triangle is 160, one side is 102, and the opposite angle 41°6′ : find the radii of the circumcircle and the excircle opposite to the given angle.

179. Prove, with the usual notation, that $AI^2 = bc\frac{s-a}{s}$; and find similar expressions for BI^2, CI^2.

180. Show that $AI_1^2 = \frac{sbc}{s-a}$.

*181. Prove that $AI \cdot AI_1 = AI_2 \cdot AI_3 = bc,$

and $BI_1 \cdot CI_2 \cdot AI_3 = CI_1 \cdot AI_2 \cdot BI_3 = abc.$

182. In a triangle ABC, right-angled at A, prove that

$$2r = AB + AC - BC, \quad 2r_1 = AB + AC + BC.$$

(See *Geometry*, Exer. 164.)

183. With the usual notation for a triangle, prove the following identities :—

(i) $\frac{1}{2Rr} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}$; (ii) $\frac{1}{2Rr_1} = \frac{1}{ab} + \frac{1}{ca} - \frac{1}{bc}$;

(iii) $rr_1r_2r_3 = \Delta^2$; (iv) $\frac{c^2+a^2-ca\cos B}{a\sin A+b\sin B+c\sin C} = \frac{b}{2\sin B}$.

184. Prove that $\cot\frac{B}{2}\cot\frac{C}{2} - 1 = \frac{2a}{b+c-a}$,

and $1 - \tan\frac{B}{2}\tan\frac{C}{2} = \frac{2a}{b+c+a}$.

*185. Prove that $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, with similar values of $\sin\frac{B}{2}$, $\sin\frac{C}{2}$.

Hence deduce $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$.

186. Given $a=5$, $b=3$, $C=60°30'$, prove that

$$A = 82°57', \quad B = 36°33'.$$

187. Given $a=1$, $b=7$, $c=\sqrt{56}$, show that $C = 115°22'$.

188. Calculate the angles of a triangle when—

(i) $a=2$, $b=3$, $c=4$; (ii) $3a=7c$, $B=45°$.

189. If d_1, d_2, d_3 are the diameters of the excircles of a triangle ABC, prove that

$$d_1 d_2 + d_2 d_3 + d_3 d_1 = (a+b+c)^2.$$

190. Three circles of radii a, b, c, touch each other externally two and two: if r is the radius of the incircle of the triangle formed by their centres, show that

$$\frac{1}{r^2} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}.$$

191. If p_1, p_2, p_3 are the perpendiculars from the vertices A B, C, to the opposite sides of the triangle ABC, prove that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3};$$

and $$p_1 = \frac{2r_2 r_3}{r_2+r_3} = \frac{2r r_1}{r_1-r},$$

with similar expressions for p_2 and p_3.

192. If in any triangle a, b, c are in A.P., then

$\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., and $\cot\frac{A}{2}\cot\frac{C}{2} = 3$.

193. Verify the following formulæ with the usual notation:—

(i) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;

(ii) $r_1\cos\frac{A}{2} = a\cos\frac{B}{2}\cos\frac{C}{2}$, $r_2\cos\frac{B}{2} = b\cos\frac{C}{2}\cos\frac{A}{2}$;

(iii) $r_1 - r = a\tan\frac{A}{2}$, $r_2 + r_3 = a\cot\frac{A}{2}$;

(iv) $r_1 = r\cot\frac{B}{2}\cot\frac{C}{2}$, $r_2 = r\cot\frac{C}{2}\cot\frac{A}{2}$, $r_3 = r\cot\frac{A}{2}\cot\frac{B}{2}$;

(v) $r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \Delta$.

194. In any triangle prove that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},$$

and $$\sin A\cos^2\frac{B}{2} + \sin B\cos^2\frac{A}{2} = \frac{1}{2}(\sin A + \sin B + \sin C).$$

195. Find, in terms of the sides of the triangle ABC, the values of $4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, $4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

CHAPTER XV.

SOLUTION OF TRIANGLES AND APPLICATIONS.

§ 34. SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

It is well-known that there are four cases in which the identical equality of two triangles may be asserted: see *Geometry*, § 1, [4, 5, 6, 7]. It follows that if the data of any one of these propositions are given and triangles are drawn in accordance with them, the triangles will be exact copies one of the other, and their other parts, which are not given in the data, will be respectively identical. Thus if, as in [4], the two sides of a triangle are given in length and the included angle is known, the triangle if constructed geometrically will have the remaining parts *always* the same. When these remaining parts are calculated by means of algebraical formulæ involving the data, the *triangle* is said to be *solved*; and we see that, if the data are consistent, each of the remaining parts should be determined in one way only. Similar remarks apply when the triangle is calculated from the data of [5] and [6]; in the case of [7] there may be an ambiguity.

For the data to be consistent, one angle or the sum of two angles must not be greater than two right angles; and one side must not be equal to or greater than the sum of the other two sides. In the case of theorem [7], the circle and the line employed in the geometrical construction must meet in one point at least. Algebraically considered, the consistency of the data with each other will require any quantity whose square root has to be taken to be positive: trigonometrically, we must have any sine or cosine occurring in the calculation less than unity.

When an angle is determined from its sine, there may be more than one solution, as such an angle can take either of two supplementary values. In this case, algebraically, both signs attached to a real square root will be found to be applicable; geometrically, both points of intersection of a circle and a line will give possible solutions. When an angle, however, can be determined from its cosine or tangent, there will be no ambiguity.

We shall now consider the solution of triangles in the order of the propositions referred to above.

I. *Given two sides and the included angle, to solve the triangle.*

Let b, c, A be given, b being greater than c.

If the numbers b, c are small, and if, in particular, A is one of the special angles of § 27, the third side may be calculated from $a^2 = b^2 + c^2 - 2bc \cos A$; the remaining angles may be found by means of the Cosine Formulæ from the Table of Natural Cosines.

But the following method, in which logarithms can be readily employed, is more generally useful.

We have $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$, [§ 33, V]

so that $L\tan \frac{B-C}{2} = \log(b-c) - \log(b+c) + L\cot \frac{A}{2}$.

From this $\frac{B-C}{2}$ is found by help of the Table of Logarithmic Tangents: also $\frac{B+C}{2}$, being $90° - \frac{A}{2}$, is known.

Hence, by addition and subtraction, we at once get B and C.

The third side is then found from the formula

$$a \sin B = b \sin A,$$

which gives $\log a = \log b + L \sin A - L \sin B$.

II. *Given three sides, to solve the triangle.*

If the numbers a, b, c are small, the angles may be obtained from the Table of Natural Cosines by means of $\cos A = (b^2 + c^2 - a^2) \div 2bc$ and two similar formulæ.

The following formulæ, suited to logarithmic calculation, are of more general use:—

$$\tan^2 \frac{A}{2} = \frac{(s-b)(s-c)}{(s-a)}, \quad \tan^2 \frac{B}{2} = \frac{(s-c)(s-a)}{(s-b)},$$

$$\tan^2 \frac{C}{2} = \frac{(s-a)(s-b)}{(s-c)}. \qquad [\S\ 33, IV]$$

Only four logarithms, those of s, $s-a$, $s-b$, $s-c$, will be required to obtain all the angles. The third angle, C, may be found from $C = 180° - A - B$; or, this result may be employed in verifying the value obtained from the formula for $\tan^2 \frac{C}{2}$.

The values of $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, etc., found in Exam. 6 and Exer. 185 of § 33, may also be employed. In determining all the angles, they will require the additional logarithms of a, b, c.

III. *Given two angles and one side, to solve the triangle.*

Supposing B, C, b given, we have $A = 180° - B - C$.

Then $a \sin B = b \sin A$ gives

$$\log a = \log b + L \sin A - L \sin B,$$

whence a can be found from the Table of Logarithms of Numbers. Similarly, c is obtained from

$$\log c = \log b + L \sin C - L \sin B.$$

IV. *Given two sides and the angle opposite to one of them, to solve the triangle.*

Let b, c and B, opposite to the side b, be given.

The formula $\sin C = \dfrac{c \sin B}{b}$ gives

$$L \sin C = \log c + L \sin B - \log b.$$

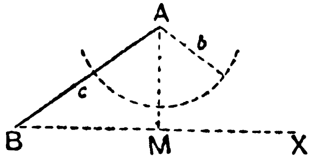
Thus C is determined; and then A is found from

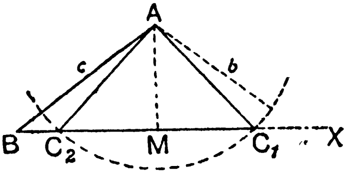
$$A = 180° - B - C.$$

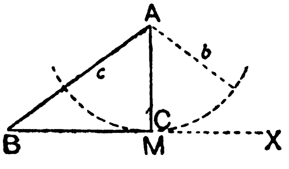
The third side a may now be found from the Rule of Sines.

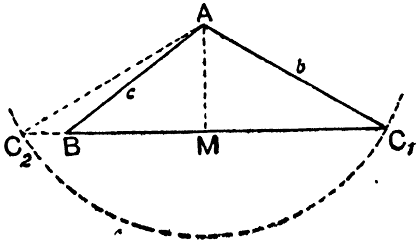
As the sine of an angle is the same as that of its supplement, we may take for C either the value furnished by the Tables or its supplementary value. Then A will have two values, corresponding to these values of C; and, if both are applicable, the side a will also have two corresponding values, so that two triangles may exist with the three given parts. Hence this case is called the *ambiguous case* in the solution of triangles.

We give below a geometrical investigation of the problem, finding out when the solution is *unique* and when *ambiguous*.









Make an angle XBA = B, and on one arm mark off BA $= c$, the length of the given *adjacent* side; with centre A and radius $= b$, the length of the given *opposite* side, describe a circle. The points, if any, in which this circle cuts the other arm BX, will give the possible positions of the third vertex C.

Draw AM perp. to BX; then $\frac{\text{MA}}{\text{AB}} = \sin \text{B}$, or MA $= c \sin \text{B}$.

(i) If b (AC), the radius of the ○ described, is less than $c \sin \text{B}$ (MA), the ⊙ *does not meet* BX *at all: no triangle can exist,* then, with the given parts.

(ii) If b (AC) $= c \sin \text{B}$ (MA), the ⊙ *touches* BX *at one point, viz.,* M itself: in this case, *only one triangle exists* with the given parts, and it is right-angled at C (M).

(iii) If $b > c \sin \text{B}$, but $< c$ (AB), the ○ *cuts* BX *in two points,* C_1 and C_2, *on the same side of* B: two triangles, ABC_1 and ABC_2, now exist, *both having the three given parts b, c, B.*

(iv) If b is not only $> c \sin \text{B}$, but also $> c$, the ○ *cuts* BX *in two points,* C_1 and C_2, *on opposite sides of* B: of the two triangles ABC_1 and ABC_2 thus formed, ABC_1 *alone has the given parts b, c,* B; the other, ABC_2, has the angle ABC_2 equal *not to* B but to its supplement, and therefore *does not satisfy the data.* Thus in this case there is *only one* solution.

If $b = c$, C_2 coincides with B, and *only one triangle,* an isosceles one, is formed with the given parts.

We, therefore, conclude that there will be *real ambiguity only when b* (AC) *is* $> c \sin \text{B}$ (MA) *and* $< c$ (AB); in other cases, either the triangle is impossible or only one triangle can be formed from the data or found to satisfy them.

Examples. 1. Given $c = 65$, $a = 25$, $\text{B} = 75^\circ 12'$, solve the triangle.

We have to employ the following formulæ—

$$\tan \frac{\text{C}-\text{A}}{2} = \frac{c-a}{c+a} \cot \frac{\text{B}}{2}, \quad \frac{b}{a} = \frac{\sin \text{B}}{\sin \text{A}}.$$

Now $\frac{c-a}{c+a} \cot \frac{\text{B}}{2} = \frac{4}{9} \cot 37^\circ\ 36'$;

$$\therefore \quad \text{L}\tan \frac{\text{C}-\text{A}}{2} = \log 4 + \text{L} \cot 37^\circ\ 36' - \log 9$$

$$= \cdot 6021 + 10\cdot 1135 - \cdot 9542 = 9\cdot 7614.$$

Hence $\frac{\text{C}-\text{A}}{2} = 30^\circ$; also, $\frac{\text{C}+\text{A}}{2} = 90^\circ - \frac{\text{B}}{2} = 52^\circ\ 24'$;

$$\therefore \quad \text{C} = 82^\circ\ 24', \quad \text{A} = 22^\circ\ 24'.$$

$$\begin{aligned}\text{Again } \log b &= \log a + \mathrm{L}\sin \mathrm{B} - \mathrm{L}\sin \mathrm{A}\\ &= \log 25 + \mathrm{L}\sin 75^\circ\, 12' - \mathrm{L}\sin 22^\circ\, 24'\\ &= 1{\cdot}3979 + 9{\cdot}9853 - 9{\cdot}5810 = 1{\cdot}8022.\end{aligned}$$

$\therefore\ b = 63{\cdot}42$, or $63\frac{1}{2}$ very nearly.

2. Given $a = 2502$, $b = 4263$, $c = 5617$, find all the angles.

$$\begin{array}{rl} 2502 = a & s - a = 3689\\ 4263 = b & s - b = 1928\\ \underline{5617 = c} & \underline{s - c = \ \ 574}\\ 2)\underline{12382\ \ } & s = 6191\\ 6191 = s & \end{array}$$

Now $\tan\frac{\mathrm{A}}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$;

$$\begin{aligned}\therefore\quad \mathrm{L}\tan\frac{\mathrm{A}}{2} &= 10 + \frac{1}{2}\{\log 1928 + \log 574 - \log 6191 - \log 3689\}\\ &= \ \ 10\\ &\quad + 1{\cdot}6426 - 1{\cdot}8959\\ &\quad + 1{\cdot}3795 - 1{\cdot}7835\\ &= \ \ 13{\cdot}0221\\ &\quad - 3{\cdot}6794 = 9{\cdot}3427.\end{aligned}$$

$\therefore\quad \frac{\mathrm{A}}{2} = 12^\circ\, 25'$, and $\mathrm{A} = 24^\circ\, 50'$.

$$\begin{aligned}\text{So also } \mathrm{L}\tan\frac{\mathrm{B}}{2} &= 10 + \frac{1}{2}\{\log 574 + \log 3689 - \log 6191 - \log 1928\}\\ &= \ \ 10\\ &\quad + 1{\cdot}3795 - 1{\cdot}8959\\ &\quad + 1{\cdot}7835 - 1{\cdot}6426\\ &= \ \ 13{\cdot}1630\\ &\quad - 3{\cdot}5385 = 9{\cdot}6245.\end{aligned}$$

$\therefore\quad \frac{\mathrm{B}}{2} = 22^\circ\, 50'$, and $\mathrm{B} = 45^\circ\, 40'$.

Hence $\mathrm{C} = 180^\circ - 70^\circ\, 30' = 109^\circ 30'$.

We have $\mathrm{L}\tan\frac{\mathrm{C}}{2} = 10 + \frac{1}{2}\{\log 3689 + \log 1928 - \log 6191 - \log 574\}$;

this gives $\mathrm{L}\tan\frac{\mathrm{C}}{2} = 10{\cdot}1507$, whence $\frac{\mathrm{C}}{2} = 54^\circ\, 45'$. Thus a useful verification of the previous work is obtained.

3. In a triangle $\mathrm{B} = 60^\circ 17'$, $\mathrm{C} = 34^\circ 33'$, $c = 149$: find the other parts.

We at once obtain $\mathrm{A} = 85^\circ 10'$.

From $\frac{a}{c} = \frac{\sin \mathrm{A}}{\sin \mathrm{C}}$,

$$\begin{aligned}\text{we get } \log a &= \log 149 + \mathrm{L}\sin 85^\circ 10' - \mathrm{L}\sin 34^\circ\, 33'\\ &= \ \ 2{\cdot}1732\\ &\quad + 9{\cdot}9985 - 9{\cdot}7537\\ &= 12{\cdot}1717 - 9{\cdot}7537 = 2{\cdot}4180.\end{aligned}$$

Hence $a = 261{\cdot}8$.

So also b may be shown to be $228{\cdot}4$.

4. Given $a=6$, $b=8$, $B=45^\circ$, find the other angles.

We have $\sin A = \frac{a}{b}\sin B = \frac{3}{4}\cdot\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{8} = \cdot 5304$;

hence $A = 32^\circ 2'$, from the Table of Natural Sines.

$\therefore\quad C = 180^\circ - (A+B) = 180^\circ - 77^\circ 2' = 102^\circ 58'$.

We could also have $A = 180^\circ - 32^\circ 2' = 147^\circ 58'$; but, in this case, $C = 180^\circ - 192^\circ 58' = -(12^\circ 58')$, which would be impossible in a real triangle. Thus there is only one solution, as given by IV above: see also § 32, Exam. 2.

5. Given $a=6$, $b=8$, $A=45^\circ$, solve the triangle.

Here $\sin B = \frac{b}{a}\sin A = \frac{4}{3}\cdot\frac{1}{\sqrt{2}} = \frac{2\sqrt{2}}{3} = \cdot 9428$;

hence $B = 70^\circ 32'$, from the Tables.

Putting $70^\circ 32' = B_1$, and its supplement $109^\circ 28' = B_2$,
we have $C_1 = 180^\circ - (A+B_1) = 64^\circ 28'$;
and $C_2 = 180^\circ - (A+B_2) = 25^\circ 32'$.

In the first case, $c_1 = \frac{a\sin C_1}{\sin A} = 6\sqrt{2}\sin 64^\circ 28'$;

$$\begin{aligned}\therefore\ \log c_1 &= \log 6 + \tfrac{1}{2}\log 2 + L\sin 64^\circ 28' - 10\\ &= \quad \cdot 7782\\ &\quad + 0\cdot 1505\\ &\quad + 9\cdot 9553 - 10 = \cdot 8840. \quad \text{Hence } c_1 = 7\cdot 656.\end{aligned}$$

In the second case, $c_2 = 6\sqrt{2}\sin 25^\circ 32'$;

$$\begin{aligned}\therefore\ \log c_2 &= \quad \cdot 7782\\ &\quad + 0\cdot 1505\\ &\quad + 9\cdot 6345 - 10 = \cdot 5632. \quad \text{Hence } c_2 = 3\cdot 658.\end{aligned}$$

Thus there exist two distinct triangles with the three given parts, in accordance with IV above. The student should check the result by drawing a tolerably accurate figure.

From the two examples above as well as the investigation in IV, it is clear that when the *given angle is opposite to the larger side* there is no ambiguity, supposing the triangle to be possible.

The following trigonometrical method may be used in place of the geometrical investigation in IV above.

The given parts being b, c, B, we have

$$\sin C = \frac{c\sin B}{b} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

(i) When $b < c\sin B$, $\sin C > 1$, and no angle exists satisfying equation (1): hence, in this case, there is no solution.

(ii) When $b = c\sin B$, $\sin C = 1$, and $C = 90^\circ$: thus there is one solution, the triangle being right-angled at C.

(iii) When $b > c\sin B$, $\sin C < 1$, and two supplementary angles exist both satisfying equation (1). Let these be denoted by C_1, C_2, the former being acute and the latter obtuse.

If now $b < c$, $B < C$, and it is possible for B to be less than C_1 as well as C_2: thus there exist two solutions in this case.

(iv) But, if further $b=c$ or $>c$, $B=C$ or $>C$, and it is not possible to take the obtuse value C_2 for C; for then $B=C_2$ or $>C_2$, and the triangle has two angles each greater than a right angle, which is impossible. Hence there exists only one solution, viz. that given by the acute value C_1 satisfying (1).

We have supposed B to be acute; if it is obtuse, only the acute value of C will be applicable, and there will be only one solution when the triangle is at all possible.

Exercises.

196. Solve the triangle ABC, given—

(i) $a=25$, $b=65$, $C=65^\circ\ 26'$;

(ii) $a=135{\cdot}8$, $b=194{\cdot}5$, $c=239{\cdot}7$;

(iii) $B=51^\circ 31'$, $C=76^\circ$, $b=98{\cdot}2$;

(iv) $B=51^\circ 31'$, $C=76^\circ$, $a=98{\cdot}2$;

(v) $a=70$, $b=82$, $A=27^\circ 49'$;

(vi) $a=70$, $b=82$, $B=27^\circ 49'$.

197. Given $3c=7b$, $A=37^\circ\ 8'$, find the other angles.

198. Given $a=723$, $b=925$, find c— when $C=35^\circ\ 16'$ and when $C=144^\circ 44'$.

199. The three sides of a triangle being as $4:5:7$, find the mean angle.

200. Given $a=3$, $b=\sqrt{3}+1$, $c=\sqrt{6}$, find all the angles.

201. Two angles of a triangle are $65^\circ 31'$, $59^\circ 29'$, and the shortest side is one mile : find the longest side.

202. Two angles of a triangle are $131^\circ 2'$, $31^\circ 2'$, and the longest side is 1000 yds : find the shortest side.

203. The three sides of a triangle are 100, $174{\cdot}6$, $203{\cdot}4$: find the greatest angle from the formula for $\tan\frac{A}{2}$, and the least from that for $\cos\frac{A}{2}$.

204. Is there any ambiguity in the solution of the triangle ABC from the following data ?—

(i) $b=40$, $c=89$, $B=30^\circ$; (ii) $a=4$, $b=4+\sqrt{80}$, $A=18^\circ$;

(iii) $a=5$, $c=7$, $A=70^\circ 17'$; (iv) $a=7$, $c=5$, $A=70^\circ 17'$.

205. Solve completely the triangle in which $b=394{\cdot}5$, $c=239{\cdot}7$, $B=47^\circ 13'$.

206. Solve completely the triangle in which $b=117$, $c=44{\cdot}01$, $C=20^\circ 37'$: given—

$\log 11{\cdot}7=1{\cdot}0682$, $\log 44{\cdot}01=1{\cdot}6436$, $\log 12{\cdot}5=1{\cdot}097$, $\log 94{\cdot}01=1{\cdot}9732$;

$L\sin 20^\circ 37'=9{\cdot}5466$, $L\sin 69^\circ 23'=9{\cdot}9712$, $L\sin 48^\circ 46'=9{\cdot}8762$.

207. In a triangle the two sides are 61, 39, and the difference of the base-angles is $35°22'$: find all the angles, given—

$$\log 22 = 1{\cdot}3424,\ \text{L} \tan 17°41' = 9{\cdot}5036,\ \text{L} \cot 34°36' = 10{\cdot}1612.$$

208. Shew how to solve a triangle, when two of the angles are given and the area or the perimeter is known.

209. Given b, c, B, as in IV, if the solution is really ambiguous prove that the difference of the two values of a is

$$2\sqrt{(b^2 - c^2 \sin^2 \text{B})}.$$

*210. In any triangle ABC given b, c, B, prove that a may be found from the quadratic equation

$$a^2 - 2ac \cos \text{B} + c^2 - b^2 = 0.$$

Discuss the roots of this quadratic and obtain the results derived in IV above.

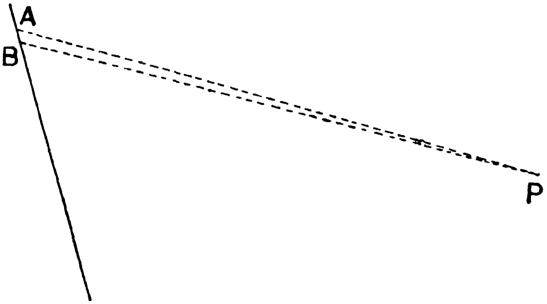
*211. When a, b, A are the given parts of a triangle ABC, and the solution is really ambiguous, prove that the two values c_1, c_2, of the third side, are connected by the relations $c_1 + c_2 = 2b \cos \text{A}$, $c_1 c_2 = b^2 - a^2$. Give also a geometrical proof of these results.

§ 35. APPLICATION TO PRACTICAL AND GEOMETRICAL PROBLEMS.

I. We have already given examples of the practical solution of right-angled triangles in § 30, and shown how other triangles may be solved by resolving them into right-angled triangles. In the following examples the formulæ of oblique-angled triangles are directly applied.

In all cases it will be found that a fundamental straight line, called the *base line*, has to be measured and two angles (at the least) have to be observed. The required distances or heights are then calculated usually by the *rule of sines*, as in III, § 34. In more complicated problems more than two angles are observed; and a second base line may sometimes be required. Cases I and II of § 34 are of rare occurrence in practice; of course case IV (the ambiguous case) should never be employed.

Examples. 1. In order to ascertain the distance of an inaccessible object P on a horizontal plane, a convenient base line AB, of length 20 yds., is measured; the angles PAB, PBA are observed to be 60° and 119°20′ respectively. Prove that the distance PB is nearly $\frac{37}{44}$ of a mile.



Applying the *rule of sines* to the ΔPAB, we have

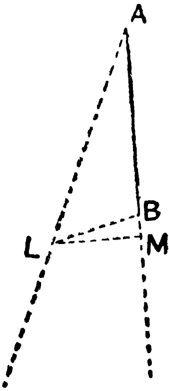
$$\frac{PB}{AB}=\frac{\sin 60^\circ}{\sin 40'}=\frac{\cdot 8660}{\cdot 0117}=\frac{8660}{117};$$

$$\therefore\ PB=\frac{8660\times 20}{117}\text{yds.}=\frac{8660\times 20}{117\times 1760}\text{ mile.}$$

Simplifying, we get $\frac{4330}{117\times 44}=\frac{37}{44}$ very nearly.

We do not employ logarithmic sines as L sin 40′ is not given by four-figure tables,—the logarithms of the sine increasing irregularly when the angle is below 3°. The angle at P can be only roughly shown.

2. A ship is steaming due south at 15 knots per hour; a lighthouse at first bears S 24° W, and 15 minutes later it is exactly WSW. Find the distance of the ship from the lighthouse at the first observation.



Here the distance between A and B, the two positions of the ship, is the base line of length $3\frac{3}{4}$ knots. If L is the position of the lighthouse,

$$\angle BAL=24^\circ,\quad \angle ABL=90^\circ+22\tfrac{1}{2}^\circ=112\tfrac{1}{2}^\circ;$$

so that $\angle ALB=180^\circ-136\frac{1}{2}^\circ=43\frac{1}{2}^\circ$.

$$\text{Thus}\ \frac{AL}{AB}=\frac{\sin 112\frac{1}{2}^\circ}{\sin 43\frac{1}{2}^\circ}=\frac{\sin 67\frac{1}{2}^\circ}{\sin 43\frac{1}{2}^\circ};$$

$$\begin{aligned}\therefore\quad \log AL &= \log 3\cdot 75+L\sin 67^\circ 30'-L\sin 43^\circ 30'\\ &= \cdot 7018,\ \text{from the Tables.}\end{aligned}$$

Hence AL = 5·032 knots = 5·8 miles nearly.

[A knot or nautical mile is almost exactly 2025 yards.]

3. In the preceding example, if the ship continues on its course find the least distance at which it will pass the lighthouse.

Drawing LM perp. to AB produced, it is seen that M is the point of the course nearest to the lighthouse.

Now $LM = AL \sin BAL$;

$$\therefore \log LM = \log AL + L \sin 24^\circ - 10$$
$$= \cdot 7018 + 9\cdot 6093 - 10$$
$$= \cdot 3111.$$
$$\because\ LM = 2\cdot 046 \text{ knots}.$$

[Here it is convenient to use not AL but its logarithm found above.]

4. From two points A and B, 1500 yds. apart, an aeroplane C is observed to have simultaneous elevations of 41°23′ and 46°32′ respectively: if the plane ABC is vertical and the vertical line through C is between A and B, find the height of the aeroplane.

See Exam. 10, § 30, to the figure of which reference is made.

Here $\angle ACB = 180^\circ - 87^\circ 55'$.

$$\text{Thus}\quad \frac{AC}{AB} = \frac{\sin 46^\circ 32'}{\sin(180^\circ - 87^\circ 55')} = \frac{\sin 46^\circ 32'}{\sin 87^\circ 55'},$$

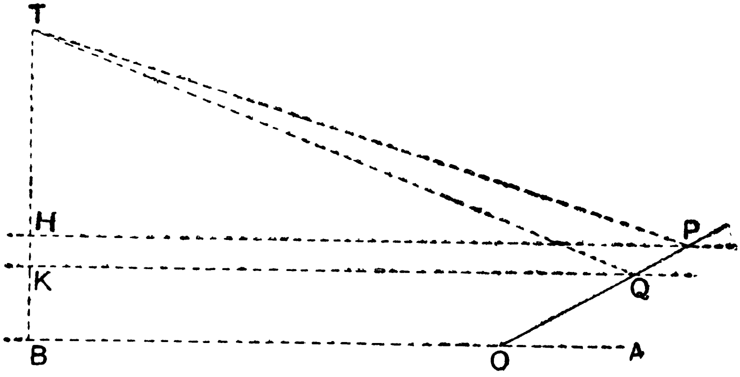
$$\text{and}\quad \frac{CD}{AC} = \sin 41^\circ 23';$$

$$\therefore CD = 1500 \sin 46^\circ 32' \times \sin 41^\circ 23' \div \sin 87^\circ 55'.$$
$$\therefore \log CD = 3\cdot 1761 + 9\cdot 8608 + 9\cdot 8202 - 9\cdot 9997 - 10 = 2\cdot 8574.$$

Hence $CD = 720\cdot 1$, or 720 yds., almost exactly.

In some problems the base line lies in a vertical plane of the figure at some inclination to the horizon : see below and Exam. 6 of § 31.

5. The elevation of the top of a hill observed from a point on an opposite slope is 18° ; after descending the slope, whose constant inclination' to the horizon is 28°30′, for a distance of 200 yds., the elevation is found to be 22°. Find the height of the hill-top above the level of the second point of observation.



Let T be the hill-top, P and Q the two points of observation on the slope OQP of constant inclination ; AOB, QK, PH horizontal lines, and THK the vertical through T.

Then $\angle TPQ = \angle TPH + \angle HPQ = \angle TPH + \angle POA$
$= 18^\circ + 28^\circ 30' = 46^\circ 30'$;
similarly $\angle TQO = \angle TQK + \angle QOA = 50^\circ 30'$;
so that $\angle QTP = \angle TQO - \angle TPQ = 4^\circ$.

Now $\dfrac{TQ}{QP} = \dfrac{\sin TPQ}{\sin QTP}$, and $TK = TQ \sin TQK$;

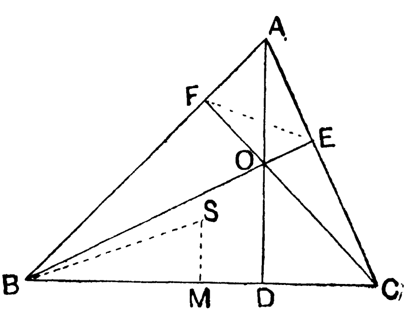
$$\therefore \quad TK = \frac{QP \sin TPQ . \sin TQK}{\sin QTP} = \frac{200 \sin 46^\circ 30' . \sin 22^\circ}{\sin 4^\circ}.$$

Thus
$$\begin{aligned} \log TK &= \log 200 + L \sin 46^\circ 30' + L \sin 22^\circ - L \sin 4^\circ - 10 \\ &= 2·3010 - 8·8436 \\ &\quad + 9·8606 - 10 \\ &\quad + 9·5736 \\ &= 21·7352 - 18·8436 = 2·8916\,; \end{aligned}$$
$\therefore TK = 779$ yds. almost exactly.

II. We shall conclude by giving some examples of the investigation of geometrical properties of triangles and polygons by means of the triangle formulæ and the properties of trigonometric functions.

6. The perpendiculars of a triangle are drawn; to find the sides of the pedal triangle, and the segments of the perpendiculars made by the orthocentre.

Let AD, BE, CF be the perps. of the Δ ABC, meeting in O, the orthocentre [XVIII, § 11]; then DEF is the pedal triangle. It is easily seen that $\angle AEF = B$, $\angle AFE = C$.



Hence, from the Δ AEF,
$$\frac{EF}{EA} = \frac{\sin EAF}{\sin AFE} = \frac{\sin A}{\sin C};$$
but $\dfrac{EA}{AB} = \cos A$,
as $\angle BEA = 90^\circ$.
$$\therefore \quad \frac{EF}{AB} = \frac{\sin A \cos A}{\sin C},$$
i. e., $EF = \dfrac{c \sin A \cos A}{\sin C}$.

As $2R = \dfrac{c}{\sin C}$ [§ 33, I],

we can also write $EF = 2R \sin A \cos A = a \cos A$.
So we shall have $FD = 2R \sin B \cos B = b \cos B$,
$DE = 2R \sin C \cos C = c \cos C$.

Again $\dfrac{AO}{AE} = \sec OAE = \sec (90^\circ - C) = \operatorname{cosec} C$;

and $\dfrac{EA}{AB} = \cos A$. $\therefore \dfrac{AO}{AB} = \cos A \operatorname{cosec} C$.

Hence $AO=\frac{c \cos A}{\sin C}=2 R \cos A.$

So also, $BO=2 R \cos B$, $CO=2 R \cos C$.

Finally $\frac{DO}{OB}= \sin OBD=\cos C$;

$\therefore$ $DO=OB \cos C=2 R \cos B \cos C$.

So also, $EO= 2 R \cos C \cos A$, $FO=2 R \cos A \cos B$.

If S is the circumcentre, we can prove that the distance of S from BC is R cos A: hence AO is double of this distance. See *Geometry*, XVIII, *Note* 4. Also, AO. OD, BO. OE, CO. OF are all equal, being $4 R \cos A \cos B \cos C$, as was to be expected.

7. To find the distance between the centres of the incircle and the circumcircle of a triangle in terms of the radii of these circles.

With the figure of Exam. 24, Misc. IV (*Geometry*), we have

$AI.\ IT=yI.Iz=Oy^2-OI^2=R^2-OI^2$;

$\therefore$ $OI^2=R^2-AI.\ IT$.

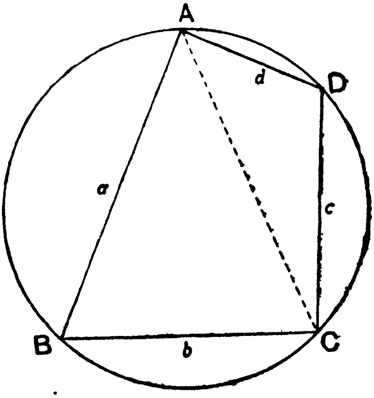
But $\frac{AI}{Ix} = \operatorname{cosec}\frac{A}{2}$, $\frac{IT}{TT_1}=\frac{CT}{TT_1}= \frac{A}{2}$;

$\therefore AI.\ IT =r \operatorname{cosec}\frac{A}{2} .\ 2R \sin\frac{A}{2} = 2 R r.$

Hence $OI^2=R^2-2Rr$.

So too we can prove $OI_1{}^2=R^2+2Rr_1$, etc.

8. A quadrilateral inscribed in a circle has consecutive sides of length a, b, c, d: to prove that its area (Q) is given by $Q=\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$, where $2s=a+b+c+d$.



Let ABCD be the cyclic quadrilateral; $AB=a$, $BC=b$, $=c$, $DA = d$; join AC.

From the Δ ABC [§ 32, II],

$AC^2=a^2+b^2-2ab \cos B$;

and from the Δ ADC,

$AC^2=c^2+d^2-2cd \cos D$.

But $\because$ $D+B=180^\circ$,

$\cos D=\cos(180^\circ-B)=-\cos B$;

$\therefore a^2+b^2-2ab \cos B=c^2+d^2+2cd \cos B$.

Hence $\cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$.

$$\therefore \sin^2 B=1-\cos^2 B=\frac{4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2}{4(ab+cd)^2}.$$

The numerator may be proved to be

$(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$;

see Hall and Knight, *Elem. Algebra*, XXVIII a, Ex. 39.

But $a+b+c-d = a+b+c+d-2d = 2(s-d)$, etc.

$$\therefore \sin^2 \text{B} = \frac{16\,(s-a)(s-b)(s-c)(s-d)}{4(ab+cd)^2};$$

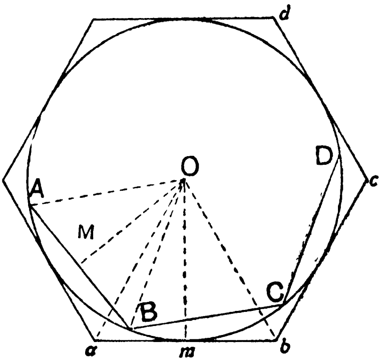
$$\therefore \sin \text{B} = \frac{2\sqrt{\{(s-a)\,(s-b)\,(s-c)\,(s-d)\}}}{ab+cd}.$$

Now $\text{Q} = \Delta\,\text{ABC} + \Delta\,\text{ADC} = \frac{1}{2}\,ab \sin \text{B} + \frac{1}{2}\,cd \sin \text{D}$
$= \frac{1}{2}\,(ab+cd) \sin \text{B}$, as $\sin \text{D} = \sin(180^\circ - \text{B}) = \sin \text{B}$.

Hence, substituting the value of sin B found above, we have $\text{Q} = \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$.

If D and A coincide, $d=0$, the quadrilateral becomes the Δ ABC, and we get the usual expression for its area.

9. To find the side and area of (i) an inscribed (ii) a circumscribed regular polygon of a given circle.



Let R be the radius of the ⨀, n the number of sides of an inscribed regular polygon ABCD...Join the centre O to A and B, draw OM perp. to AB; then AB is bisected in M and OM bisects ∠ AOB.

$$\therefore \angle\,\text{AOM} = \frac{1}{2}\,\angle\,\text{AOB}$$
$$= \frac{1}{2} \cdot \frac{360^\circ}{n},$$

as all the ∠s AOB, BOC, COD ...are equal.

Now $\text{AB} = 2\,\text{AM} = 2\frac{\text{AM}}{\text{AO}} \cdot \text{AO} = 2\,\text{AO} \sin \text{AOM}$;

thus $\text{AB} = \text{BC} = \text{CD} = 2\,\text{R} \sin \frac{180^\circ}{n}$.

And area $\text{ABCD}\ldots = n.\ \Delta\,\text{AOB} = n.\frac{1}{2}\,\text{AO.OB} \sin \text{AOB}$

$$= \frac{n}{2}\,\text{R}^2 \sin \frac{360^\circ}{n}.$$

Again, let $abcd$...be a circumscribed regular polygon of n sides; let ab touch the ⨀ in m, and join Oa, Ob, Om.

We can prove the Δs Oam, Obm congruent, and Om perp. to ab.

Hence $ab = 2am = 2\frac{am}{\text{O}m} . \text{O}m = 2\,\text{O}m \tan a\text{O}m$;

so that $ab = bc = cd \ldots = 2\,\text{R} \tan \frac{180^\circ}{n}$.

And area $abcd\ldots = n.\ \Delta\,\text{O}ab = n.\ \frac{1}{2}\,\text{O}m.ab$

$$= n.\frac{1}{2}\,\text{R}.\,2\,\text{R} \tan \frac{180^\circ}{n} = n\text{R}^2 \tan \frac{180^\circ}{n}.$$

The student will see, on consideration, that the circumference of the ⊙ is greater than the perimeter of ABCD...but less than that of *abcd*..., *i. e.*, $> n.$ AB and $< n.ab$; and that if we make n very large, both $n.$ AB and $n.ab$ will nearly approach the length of the circumference. For instance, let $n=120$; then the perimeter of the inscribed polygon

$$=120.2\text{R} \sin 1^\circ 30' = 2\ \text{R}.\ 120 \times {\cdot}0261769$$
$$=3{\cdot}141228\ (2\text{R}):$$

while the perimeter of the circumscribed polygon

$$=120.2\ \text{R} \tan 1^\circ 30' = 2\text{R}.\ 120 \times {\cdot}0261859$$
$$=3{\cdot}142308\ (2\ \text{R}).$$

Thus we find that the circumference bears to the diameter (2R) a ratio lying between 3·1412 and 3·14231; it is known that this ratio is 3·14159 to five places of decimals.

The values of the trigonometrical functions of $1^\circ 30'$ are taken from seven-figure tables; in four-figure tables both values will be given as ·0262.

Exercises.

212. From C, the top of a tower 75 ft. high, the depression of a point A on the nearer bank of a river is found to be 45°, and that of B on the farther bank to be 15° : prove that the breadth AB is 205 ft. very nearly.

213. An observer at A finds that he is in a line, bearing 30° W of N, with two distant objects P and Q; he walks a distance of 146 yds. due west to B, and finds that Q bears exactly N and P is NNE. Find the distances AP, PQ.

214. Three objects A, B, C lie on ground which is so uneven that the lines AB, BC, CA cannot be measured. An observer finds the angle ABC at B and the angle BAC at A; walking a yds. along BA produced to a point P, he also finds the angle BPC. Shew how to obtain the lengths of the sides of the triangle ABC.

215. A steamer A leaves port at 9-25 a.m. in a direction E by S, steaming at 16 knots per hour; another, B, leaves at 10 a.m. in a direction SE by E, at 18 knots per hour. Find the distance AB at 10-40 a.m.

216. A ship, sailing due East at 12 miles an hour, was observed from a lighthouse to bear 10° E of N at 0-45 p.m.; at 1-30 p.m. it was found to bear due NE. Find its distance from the lighthouse at the second observation.

217. From a ship, steaming in a direction N 15°E at 15 miles an hour, a lighthouse is observed to bear due E; 20 minutes later the lighthouse bears E 8°32′ S. Find the distance of the lighthouse from the ship at each observation.

Calculate also the least distance between the two, and find when it will be reached.

*218. At two points A and B, in a horizontal line with the base of an object whose top is P, the elevations of P are found to be α and β ($\alpha<\beta$) : prove that, if AB $= a$, the distance of the top from

is either $\dfrac{a \sin \beta}{\sin (\beta-\alpha)}$ or $\dfrac{a \sin \beta}{\sin (\beta+\alpha)}$.

[Comparing these results with those in Exer. 108, we obtain the two formulæ—

$\sin (\beta-\alpha) = \sin \beta \cos \alpha - \cos \beta \sin \alpha,$

$\sin (\beta+\alpha) = \sin \beta \cos \alpha + \cos \beta \sin \alpha.$]

*219. At two points, A and B, in a vertical line such that AB $= a$, the elevations of the top, P, of an object are α and β respectively ($\alpha<\beta$): prove that the height of P above the horizontal plane through B is $\dfrac{a \cos \alpha \sin \beta}{\sin (\beta-\alpha)}$.

220. From the terrace of a house 40 ft. high the angles of elevation of the top and depression of the base of an opposite building are observed to be 20° 32′ and 32° 20′ respectively : find the height of the building and the breadth of the roadway.

221. A rock is situated upon a uniform slope making an angle of 25° with the horizon ; at the foot of the slope the elevation of the rock is found to be 37°30′, and, after walking 300 yds. directly up the slope, it becomes 71°. Find the height of the top of the rock above the horizon.

*222. Prove that the circumcircles of the two triangles found in the *ambiguous case* are equal.

223. Verify the following expressions for the area of the pedal triangle of a given triangle ABC—

$2\mathrm{R}^2 \cos \mathrm{A} \cos \mathrm{B} \cos \mathrm{C} \; (\sin \mathrm{A} \cos \mathrm{A} + \sin \mathrm{B} \cos \mathrm{B} + \sin \mathrm{C} \cos \mathrm{C});$

$\frac{1}{2} bc \cos \mathrm{B} \cos \mathrm{C} \sin 2\mathrm{A}, \; \frac{1}{2} ca \cos \mathrm{C} \cos \mathrm{A} \sin 2\mathrm{B}, \; \frac{1}{2} ab \cos \mathrm{A} \cos \mathrm{B} \sin 2\mathrm{C}.$

224. The angle A of a triangle ABC is bisected internally and externally by the lines AH, AH$_1$, meeting BC : prove that

$$\mathrm{AH} = \frac{2\Delta}{(b+c)\sin\frac{\mathrm{A}}{2}}, \quad \mathrm{AH}_1 = \frac{2\Delta}{(c-b)\cos\frac{\mathrm{A}}{2}}, \text{ supposing AB} > \text{AC}.$$

*225. Find the distances of the incentre and the excentres of a triangle from the angular points of the triangle in terms of the sides.

226. With the usual notation for a triangle, prove that

$$a.\,\mathrm{AI}^2 + b.\,\mathrm{BI}^2 + c.\,\mathrm{CI}^2 = abc, \text{ and } \frac{1}{a.\mathrm{AI}_1^2} + \frac{1}{b.\mathrm{BI}_2^2} + \frac{1}{c.\mathrm{CI}_3^2} = \frac{1}{abc}.$$

227. Prove that $\mathrm{II}_1 = a \sec \frac{\mathrm{A}}{2}, \; \mathrm{II}_2 = b \sec \frac{\mathrm{B}}{2}, \; \mathrm{II}_3 = c \sec \frac{\mathrm{C}}{2}$;

and $\mathrm{I}_2\mathrm{I}_3 = a \operatorname{cosec} \frac{\mathrm{A}}{2}, \; \mathrm{I}_3\mathrm{I}_1 = b \operatorname{cosec} \frac{\mathrm{B}}{2}, \; \mathrm{I}_1\mathrm{I}_2 = c \operatorname{cosec} \frac{\mathrm{C}}{2}.$

228. Find the area of a plane quadrilateral whose diagonals, of lengths 61·2 and 39·8, include an angle of 94° 49′.

229. In a cyclic quadrilateral ABCD, AB $= 3$, BC $= 5$, CD $= 7$, DA $= 9$; find the area and the length of AC.

230. If a, b, c, d are the lengths of successive sides of a quadrilateral which can be inscribed in one circle and circumscribed about another, prove that its area is $(abcd)^{\frac{1}{2}}$.

231. Find the side of a regular dodecagon inscribed in a circle of radius 2 cm., and that of a regular decagon described about a circle of diameter 1″.

232. The side of a regular polygon of 24 sides is ·4″ in length: find the radii of its inscribed and circumscribed circles.

233. If a is the length of a side of a regular polygon ABCD .. of n sides, find the lengths of AC, AD, ..

*234. Obtain the area of an inscribed regular polygon of a circle of radius R in the form $n\,R^2 \sin\frac{180^\circ}{n} \cos\frac{180^\circ}{n}$. Hence prove that

$$\sin\frac{360^\circ}{n} = 2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}.$$

235. Compare the areas of a regular inscribed octagon and a regular circumscribed hexagon of a given circle.

*236. Find the perimeter of a regular polygon of 100 sides inscribed in a circle of radius 6 in., and the area of a regular circumscribed polygon of 100 sides of the same circle; given—

$$\sin 1^\circ\,48' = {\cdot}0314108,\quad \tan 1^\circ 48' = {\cdot}0314263.$$

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237. Two circles are respectively described about and inscribed in a regular polygon of n sides: prove that their radii are as 1 to $\cos\frac{180^\circ}{n}$.

238. Two regular polygons of n sides each are respectively inscribed in and described about the same circle: prove that their areas are as $\cos^2\frac{180^\circ}{n}$ to 1.

*239. In a cyclic quadrilateral ABCD, AB $= a$, BC $= b$, CD $= c$, DA $= d$: prove that

$$AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}},\quad BD = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

(See *Geometry* Exer. 390.)

240. In the exercise above, find the radius of the circumcircle.

*241. In a triangle ABC, right-angled at C, prove that

$$r_1 + r_2 = r_3 - r = c.$$

242. With the usual notation for a triangle, prove that

(i) $OI^2 + OI_1^2 + OI_2^2 + OI_3^2 = 12\,R^2$;

(ii) $\frac{OI^2}{r} + \frac{OI_1^2}{r_1} + \frac{OI_2^2}{r_2} - \frac{OI_3^2}{r_3} = \frac{2cR^2}{\Delta}$;

(iii) $\frac{OI_1^2}{r_1} + \frac{OI_2^2}{r_2} + \frac{OI_3^2}{r_3} - \frac{OI^2}{r} = 8\,R$.

(See *Geometry*, Exer. 167 and Exam. 7 solved above.)

243. A spire CD is observed from two points A and B, 50 metres apart, in the same horizontal plane as C the foot of the spire; the angles DAB, DBA, CAD are found to be 75°, 70°, 40° respectively. Find the height of the spire to the nearest decimetre.

244. A surveyor starts from A and goes 300 ft. N 20° E to B, thence 210 ft. N 15° W to C, and thence 450 ft. W 22°30′ S to D. Shew that the bearing of D from A is nearly W 40°22′ N.

MISCELLANEOUS EXAMPLES IN TRIGONOMETRY.

1. The interior angles of a polygon are in A. P.; the least angle is 120° and the common difference 5°. Find the number of sides.

2. Shew that there are eleven pairs of regular polygons which are such that the measure of an angle of one of them in degrees is equal to the measure of an angle of the other in grades.

3. If $a=\sec\theta+\operatorname{cosec}\theta\ \tan^3\theta\ (\operatorname{cosec}^2\theta+1)$,
and $b=\tan\theta-\tan^3\theta\ (\operatorname{cosec}^2\theta+1)$,
prove that $a^{\frac{2}{3}}-b^{\frac{2}{3}}=2^{\frac{2}{3}}$.

4. If $\sin\theta=a$, find the values of $\tan(5\pi-\theta)$, $\sec\left(\frac{33}{2}\pi+\theta\right)$.

5. If $\tan^3\theta=b\div a$, prove that

$$a\sec\theta+b\operatorname{cosec}\theta=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}.$$

6. If $a\sin\theta+b\cos\theta=a\operatorname{cosec}\theta+b\sec\theta$, prove that each expression is equal to

$$\pm(a^{\frac{2}{3}}-b^{\frac{2}{3}})(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}.$$

7. In a triangle ABC, right-angled at C, D and E are the points of trisection of AC, and it is found that AB+AD=CB+CD: prove that ∠ABC=36° 52′ nearly.

8. Prove the following identities :—

(i) $\dfrac{\sin^2A+m\cos A-m\cos^2A}{\sin A\ \{1+(m+1)\cos A\}}=\dfrac{\sin A}{1+\cos A}$;

(ii) $\left\{\dfrac{\tan A-\sec A+1}{\tan A+\sec A+1}\right\}^2=\dfrac{(\sin A-1)(\cos A-1)}{(\sin A+1))\cos A+1)}$;

(iii) $\sin^2\theta\{(a-b)^2\cos^2\theta-a^2\}^2+\cos^2\theta\{(a-b)^2\sin^2\theta-b^2\}^2$
$=(a^2\sin^2\theta+b^2\cos^2\theta)^2$.

9. Solve the equation
$(4-\sqrt{3})(\sec\theta+\operatorname{cosec}\theta)=4(\tan\theta\sin\theta+\cot\theta\cos\theta)$.

10. Prove that the equation $9\tan\theta-2\cos^2\theta$ is satisfied by

$$\tan\theta=\frac{1}{\sqrt[3]{3}}-\frac{1}{\sqrt[3]{9}}.$$

11. The triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are of equal area and have $\cot A_1+\cot A_2+\cot A_3=0$: prove that

$$a_1^2+a_2^2+a_3^2=b_1^2+c_1^2+b_2^2+c_2^2+b_3^2+c_3^2.$$

*12. With regard to any triangle ABC prove that

$$a\cos\theta=b\cos(C-\theta)+c\cos(B+\theta),$$

and give a geometrical illustration.

13. If AD is the median from A in the triangle ABC and c is $>b$, prove that $\tan ADC=\dfrac{4\Delta}{c^2-b^2}$.

14. If P divides the side BC of a triangle ABC so that BP : PC $=m:n$, shew that

$$AP^2=\frac{m^2b^2+n^2c^2+2mn\,bc\cos A}{(m+n)^2}.$$

15. In a triangle ABC, $c^2-b^2=ab$; prove that

$$\cos B=\frac{a+b}{2c},\quad \cos C=\frac{a-b}{2b},$$

and shew that $\angle C=2\angle B$.

16. In any triangle ABC prove that

$$(a+b+c)(\cos A+\cos B+\cos C-1)=2a\sin B\sin C,$$

and $a^2\cos B+b^2\cos C+c^2\cos A$

$$=ab+bc+ca-(ab\cos A+bc\cos B+ca\cos C).$$

17. In a triangle sin A, sin B, sin C are in H. P.; prove that $\sin^2\frac{1}{2}A$, $\sin^2\frac{1}{2}B$, $\sin^2\frac{1}{2}C$ are also in H. P.

18. The cotangents of the half-angles of a triangle are in A.P.; prove that the sines of the angles are also in A. P.

*19. For any triangle prove that

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},\quad c=(a+b)\sin\frac{C}{2}\sec\frac{A-B}{2}.$$

20. Apply the formula in the preceding example to solve a triangle in which $a=15\cdot8$, $b=10$, $C=70^\circ42'$:

given log $58=1\cdot7634$, log $258=2\cdot4116$,
L tan $54^\circ39'=10\cdot1491$, L sin $35^\circ21'=9\cdot7623$,
L tan $17^\circ35'=9\cdot5009$, L sec $17^\circ35'=10\cdot0208$, log $1566=3\cdot1947$.

21. Given $a=257$, $b=341$, $\angle A=33^\circ14'$, solve the triangle:
data – log $257=2\cdot4099$, log $341=2\cdot5328$, log $4616=3\cdot6643$;
Lsin$33^\circ12'=9\cdot7384$, diff. for $2'=4$; L sin $79^\circ53'=9\cdot9932$;
Lsin$46^\circ36'=9\cdot8613$, diff. for $3'=4$; L sin $13^\circ25'=9\cdot3655$;
log $1087=3\cdot0366$.

22. In a circle of diameter 125 a triangle whose sides b, c are respectively 117, 35, is inscribed: prove that the third side a is either 100 or $124\frac{16}{25}$.

23. The area of a triangle ABC, right-angled at C, is $\frac{1}{4}c^2 \sin 2A$ or $\frac{1}{4}c^2 \sin 2B$.

24. One angle of a triangle is 60°, the area is $10\sqrt{3}$ and the perimeter is 20: determine the sides.

25. A workman is told to make a triangular enclosure of sides 50, 21, 41 yards severally; having made the first side one yard too long, find what length he must give the other two sides to enclose the same area with the same length of fencing.

26. The bisectors of the angles A, B, C meet the opposite sides in L, M, N: prove that

$$\frac{\Delta \text{LMN}}{\Delta \text{ABC}} = \frac{2abc}{(b+c)(c+a)(a+b)}.$$

27. The angular elevation of a steeple at a place due south of it is 45°; at another place, due west of the former and at a distance a from it, the elevation is 30°. Find the height of the steeple.

28. The sides of a quadrilateral of area Q are a, b, c, d in order, and the diagonals include an angle θ: prove that

$$Q = \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta.$$

29. In a triangle ABC, right-angled at A, AD is perpendicular to BC: if r_1, r_2, r_3 are the radii of the inscribed circles of the triangles ABC, ABD, ACD, then $r_1 + r_2 + r_3 = \text{AD}$.

30. A chord PQ of the circumcircle of a triangle ABC is drawn through the excentre I_1: prove that

$$I_1P \cdot I_1Q = 2R \cdot r_1.$$

*31. S is the circumcentre and O the orthocentre of a triangle ABC: prove that the rectangle of the segments of a chord of the circumcircle passing through O is

$$8R^2 \cos A \cos B \cos C,$$

and that
$$SO^2 = R^2(1 - 8\cos A \cos B \cos C).$$

ANSWERS (*GEOMETRY*).

EXERCISES.

1. 1·3. **2.** 2·1. **6.** $\frac{467}{100}, \frac{88}{100}; \frac{879}{100}, \frac{1}{32}$.

9. 54; $6\sqrt{3}$. **22.** $\frac{34}{90}, \frac{119}{90}$. **23.** $\frac{87}{80}, \frac{87}{90}$.

28. 12, 18, 27. **33.** ·92; ·64 $(1+\frac{1}{4}\sqrt{3})$. **35.** 2·9.

48. ·66 sq. in. **49.** 19·6 mm. **50.** $\frac{3}{2}\sqrt{3}$.

53. 17·3, 10·7. **65.** 10·5; 6·6 or $\frac{1}{4}\sqrt{689}$. **66.** 4·9″.

67. 1·7 = $(\sqrt{3})$, 4 = $(7 \div \sqrt{3})$.

68. Nearly 24 ft. from the base of the column.

76. Not greater than thrice the radius of the circle.

86. $\frac{2}{5}$ and $\frac{2}{3}$ towards the centre of the smaller circle.

108. 1:2. **109.** ·7 = $(\cdot 4\sqrt{3})$, 1·4 = $(\cdot 8\sqrt{3})$. **110.** $\sqrt{2} : \sqrt{3}$.

111. $2 : \sqrt{3}$. **112.** 1·3 = $(\cdot 75\sqrt{3})$. **116.** 36°, 72°, 72°.

117. 1·4 in; 3·4 sq. in. **126.** 8·8 in; 67 and 32 mm.

127. $1{\cdot}2\sqrt{3} = 2$ nearly. **129.** $\sqrt{\{d^2 - (a \pm b)^2\}}$.

139. $\frac{89}{100}$. **143.** $6\frac{2}{3}$ cm. **144.** ·58″, ·7″.

145. $(3 \pm 2\sqrt{2})r$. **146.** $2 \pm \frac{1}{2}\sqrt{6}$. **147.** 2, 10.

148. $1\frac{1}{2}$ or 6. **149.** $6\frac{1}{2}$, $26\frac{1}{2}$. **152.** 4; 68 nearly.

160. 32·5; 3, etc. **161.** 1·65, 1·35, 5·45, 3.

194. $180° - 2\alpha$, $180° - 2\beta$, $180° - 2\gamma$, when the Δ is acute-angled;
$2\alpha - 180°$, 2β, 2γ, when α is an obtuse angle.

211. (ii) The smaller part is 7·5′ or 11·25′.

212. 15,20 ; 3, 4. **213.** $2\frac{1}{3}$, $3\frac{1}{2}$. **219.** 3 : 11.

220. 1 : 2 : 1. **221.** 1 : 4. **223.** PC = $\frac{2}{5}$ BC.

227. $61\frac{5}{7}$; $142\frac{2}{19}$. **243.** ·5″ ; 2·25″. **248.** 4·6; 2 nearly.

251. 2·7″. **252.** 5·8 nearly. **255.** $\frac{1}{3}(x_1 + x_2 + x_3)$.

258. $12\frac{7}{8}$, $15\frac{1}{3}$. **260.** ·5″. **264.** $650\frac{1}{4}$ ft.

265. 5 : 4. **288.** No.

296. 2·304, ·196, ·672. **297.** 13·3 mm. **298.** $3 \pm \frac{1}{2}(3\sqrt{3})$.

310. ·78, ·77, ·76. **312.** $1\frac{5}{7}$, 12.

313. 9 mm. = $(4\sqrt{3} - 6)$ cm. **316.** 9,10 mm.

317. ·54″. **337.** $21\frac{1}{4}$. **338.** 7mm.

340. 3:1. **341.** 3:8. **345.** 4 : 5.

346. 3 : 1. **354.** 1·4″. **358.** 9·9.

363. 3:4. **364.** $2 + \sqrt{2} : 4$.

369. $29\frac{1}{2}$. **374.** 26·7. **375.** 31 nearly.
424. On the perpendicular from the centre at a distance of $\frac{15}{32}$ in. **438.** 17·1, 1·9. **439.** 9 : 4.

Miscellaneous Examples.

I. **1.** $24\frac{1}{2}$, $26\frac{1}{2}$. **2.** 11·2 mm. **5.** 98·4 yds.
II. **1.** 32 or 25 mm. **19.** $2(\sqrt{2}-1)r$, $2(2-\sqrt{3})r$.
20. 6·278, 6·297.
III. **2.** 8·7. **12.** No; C lies on a circle.
IV. **2.** $1 : mn - n + 1$. **3.** 1·64″. **9.** ⊙ of diameter 2·1.
V. **1.** 17·5. **14.** 2·7″, 3·4″. **15.** 1 : 10.
VI. **10.** $2 \pm \sqrt{3} : 1$.
38. Radius $= \sqrt{AH \times HD} = \sqrt{BH \times HE} = \sqrt{CH \times HF}$.

ANSWERS (*TRIGONOMETRY*).

EXERCISES.

1. $87°27'$, $37°\ 26'\ 39''$; $79°\ 49'$, $127°\ 59'\ 48''$.

2. $78°\ 45'$, $101°\ 15'$. **3.** $9°\ 56'\ 24''$; 26.

4. $90\frac{8}{4}$, $55\frac{1}{2}$, $33\frac{3}{4}$. **5.** $96°\ 23'\ 42''$, $144°\ 35'\ 33''$.

6. 30, 66, 108, 156. **7.** Com. Diff. $= 11\frac{3}{7}$.

8. 900, 450, 5100. **9.** $325°$. **10.** $(8\frac{56}{88})'$.

11. $435\frac{3}{4}$ miles. **12.** $138\frac{3}{4}$ very nearly.

15. $17^g\ 18^{`}\ 75^{``}$; $111^{\cdot}\ 11\ldots g$; $11^g\ 30^{`}$ nearly.

16. ·250125; $37°\ 49'\ 13''$ nearly. **17.** 22 and 11.

21. Complementary; equal. **25.** sine $= \frac{21}{29}$, etc.

26. $53°\ 8'$; ·8, ·75. **28.** 3·276 sq. in.; 1·848, 3·548.

29. 16·73 ft. **30.** $50°54'$. **31.** 207·5 ft.

32. 7·28 metres. **33.** $31°\ 4'$. **34.** ·75 cosec 51°.

35. (cosec 36°) cm. = 17 mm. nearly. **37.** $38°\ 40'$; $27°\ 50'$.

40. 119°; $114°\ 45'$. **44.** $\sin 105° = \frac{1}{4}(\sqrt{6} + \sqrt{2})$, etc.

48. $-2 \div \sqrt{3}$. **50.** $\sqrt{3}$. **54.** 135°.

57. 8·65. **58.** 7·45. **62.** $\sqrt{2} + 1$.

63. $\cos A = \pm\sqrt{(1 - s^2)}$, $\tan A = \pm s \div \sqrt{(1 - s^2)}$, etc.

64. $\sin A = \pm\sqrt{(2v - v^2)}$, $\cos A = 1 - v$, etc.

65. (i) 90° or 180°; (ii) 90°.

66. $\sec\theta \operatorname{cosec}\theta$; $-\sin^2 A$; $103 - 78\cos\theta + 6\sin\theta$;
$4(\cos^2\theta - \sin^2\theta)^{-1}$.

67. $\frac{2m}{m^2 + 1}$, $\frac{m^2 - 1}{2m}$, etc. **68.** $\frac{a^2 + 2ab}{2b(a+b)}$, $\frac{a^2 + 2ab + 2b^2}{2b(a+b)}$.

71. $6\tan^2 A$. **72.** 0, $\frac{a^2 - b^2}{2ab}$, $\frac{b(2a^2 + b^2)}{(a^2 + b^2)^{\frac{3}{2}}}$.

86. (i) $A = 90°$ or $30°$, $B = 60°$ or $0°$; (ii) $A = 90°$, $B = 30°$.

87. (i) $35°28'$, (ii) $21°42'$, (iii) $22°38'$, (iv) $127°8'$.

88. 30° or 150°. **89.** 45°. **90.** $\tan\theta = 2$ or $\frac{1}{2}$.

91. 135°. **92.** 60°. **93.** 30°, 150°, 45°.

94. $\sin\theta = \pm\frac{1}{4}\sqrt{(10 \pm 2\sqrt{5})}$. **95.** $\tan\theta = 2 \pm \sqrt{3}$.

96. 90°. **97.** (i) $45°$, $67\frac{1}{2}°$; $157\frac{1}{2}°$;
(ii) $45°$, $135°$; $60°$, $120°$; (iii) $78°42'$, $116°\ 34'$.

98. $\frac{24}{25}$, $\frac{15}{17}$. **99.** (i) $x^2+y^2=a^2+b^2$;

(ii) $c^2\{(a-a')^2+(b-b')^2\}=(ab'-a'b)^2$;

(iii) $m^{\frac{4}{3}}n^{\frac{2}{3}}+m^{\frac{2}{3}}n^{\frac{4}{3}}=1$.

100. $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$.

101. ·6483, ·3333, ·1608, 1·2006.

102. 24°30′, 134°21′, 58°42′. **103.** 43°43′; 73°47′.

104. $A=30°42'$, $B=59°18'$, $b=40·41$;
$A=60°57'$, $B=29°3'$, $c=51·48$;
$B=64°20'$, $a=47·65$, $b=99·15$;
$A=19°35'$, $c=10·616$, $a=3·558$.

105. 29·39. **106.** $r=3·732$, $R=3·864$.

109. 47·71, 95·42. **110.** 204·9. **111.** 101·8.

112. 25, 102 nearly.

113. $\frac{1}{4}a(\sqrt{6}+\sqrt{2}-2)$, $\frac{1}{4}a(2\sqrt{3}-\sqrt{6}+\sqrt{2})$.

114. 82·85. **115.** 14·73.

116. $\frac{1}{2}d\sec 63°$, $\frac{1}{2}d\tan 63°$.

117. 57·83, 26·59. **118.** $\frac{AB(\cot 48°12'\pm\cot 60°)}{\cot 48°12'(\cot 51°+\cot 60°)}$.

119. 221·7, ·0029. **120.** $-\frac{1}{8}$, $\frac{1}{4}$, -3. **121.** -2, -1, 7, -2.

122. 9·994, 9·2101, 10·7922. **123.** 10·2718, 10·1986.

124. 41°12′, 86°40′. **125.** $\bar{1}·8706$, $\bar{1}·8161$. **129.** 20, 18.

130. 5th, 47th. **131.** $\frac{1}{5}(2q-3p+1)$, $\frac{1}{5}(p+q-1)$.

132. $2b-2a-3c$, $2a+6b+2d-4c$, $\frac{2}{5}(a+b+d)$.

133. $\frac{3125}{674}$. **134.** $\pm\frac{1+\log 4}{\log 4}$. **135.** 23·78.

136. ·002995, ·1958. **137.** ·3598, ·2363. **138.** 3·423.

139. 454 ft. **140.** 421 ft. **141.** 36 cosec 5° tan 36°25′.

142. 5 cosec 53′ sin 25°40′. **143.** $\frac{10}{3}\sqrt{3}$. **144.** 30°.

145. $C=72°13'$, $A=47°47'$; or $C=107°47'$, $A=12°13'$.

146. 7·856. **148.** $\sqrt{13}$, 1. **149.** $\sqrt{61}$; $\frac{3}{2}\sqrt{5}$; 32·7 nearly.

151. ·84; 35·8; 1309·8; 224·2. **152.** 75.

153. $32\frac{4}{13}$. **174.** (i) 1201, 24; (ii) $1002\frac{1}{2}$, 18.

177. 25, 84, 101. **178.** $77\frac{1}{2}$; 59 nearly.

188. (i) 28°56′, 46°34′, 104°30′; (ii) $A=111\frac{1}{2}°$, $C=23\frac{1}{2}°$.

195. $\frac{1}{2}(a+b+c)$, $\frac{1}{2}(-a+b+c)$.

196. (i) $A = 22^\circ 36'$, $B = 91^\circ 58'$, $c = 59{\cdot}16$; (ii) $34^\circ 30'$, $54^\circ 12'$, $91^\circ 18'$;
(iii) $a = 99{\cdot}52$, $c = 121{\cdot}8$; (iv) $b = 96{\cdot}9$, $c = 120{\cdot}1$;
(v) $B = 33^\circ 8'$ or $142^\circ 52'$, $C = 119^\circ 3'$ or $9^\circ 9'$, $c = 153{\cdot}7$ or 27 95;
(vi) $A = 23^\circ 28'$, $C = 128^\circ 43'$, $c = 137{\cdot}1$.

197. $121^\circ 25'$, $21^\circ 27'$. **198.** 541·8, 1571. **199.** $22^\circ 12'$.

201. 5865 ft. **202.** 408·3 yds. **205.** $26^\circ 29'$, $106^\circ 18'$; $a = 516$.

206. $B = 69^\circ 23'$ or $110^\circ 37'$, $A = 90^\circ$ or $48^\circ 46'$, $a = 125$ or 94·01.

207. $69^\circ 12'$, $73^\circ 5'$, $37^\circ 43'$. **213.** 170, 122.

215. 10 knots nearly.

216. $15\frac{9}{20}$ miles. **217.** 33·48, 32·55; 8·666 at 34 m. 40s.

220. 54·98, 63·19. **221.** 238 yds. **228.** 2428.

229. 106·5; 7·24. **231.** 1·035 cm., ·325″. **232.** 1·532, 1·519.

235. $4 : 3\sqrt{2}$. **236.** 3·14108, 3·14263 ft. **243.** 527.

Miscellaneous Examples.

1. 9. **4.** $\mp a \div \sqrt{(1 - a^2)}$, $-1 \div a$.

9 30°, 60°, 135°. **20.** $A = 72^\circ 14'$, $B = 37^\circ 4'$, $c = 15{\cdot}66$.

21. $B = 46^\circ 39'$ or $133^\circ 21'$, $C = 100^\circ 7'$ or $13^\circ 25'$,
$c = 461{\cdot}6$ or $108{\cdot}7$.

24 7, 8, 5. **25.** 35, 26. **27.** $a \div \sqrt{2}$

FOUR-FIGURE LOGARITHMS

AND

TRIGONOMETRICAL FUNCTIONS

NATURAL SINES.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	13
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	13
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	4	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0

NATURAL TANGENTS.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
45	1'0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1'0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1'0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1'1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1'1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1'1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1'2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1'2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1'3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1'3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1'4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1'4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1'5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1'6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1'6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1'7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1'8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1'8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1'9626	9711	9797	9883	9970	$\overline{0}057$	$\overline{0}145$	$\overline{0}233$	$\overline{0}323$	$\overline{0}413$	15	29	44	58	73
64	2'0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2'1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2'2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2'3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2'4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2'6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2'7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2'9042	9208	9375	9544	9714	9887	$\overline{0}061$	$\overline{0}237$	$\overline{0}415$	$\overline{0}595$	29	58	87	116	145
72	3'0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3'2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3'4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3'7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4'0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	106	160	213	267
77	4'3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	62	124	186	248	310
78	4'7046	7453	7867	8288	8716	9152	9594	$\overline{0}045$	$\overline{0}504$	$\overline{0}970$	73	146	220	293	366
79	5'1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	87	174	262	350	438
80	5'6713	7297	7894	8502	9124	9758	$\overline{0}405$	$\overline{1}066$	$\overline{1}742$	$\overline{2}432$	*Mean* differences not sufficiently accurate.				
81	6'3138	3859	4596	5350	6122	6912	7720	8548	9395	$\overline{0}264$					
82	7'1154	2066	3002	3962	4947	5958	6996	8062	9158	$\overline{0}285$					
83	8'1443	2636	3863	5126	6427	7769	9152	$\overline{0}579$	$\overline{2}052$	$\overline{3}572$					
84	9'514	9'677	9'845	10'02	10'20	10'39	10'58	10'78	10'99	11'20					
85	11'43	11'66	11'91	12'16	12'43	12'71	13'00	13'30	13'62	13'95					
86	14'30	14'67	15'06	15'46	15'89	16'35	16'83	17'34	17'89	18'46					
87	19'08	19'74	20'45	21'20	22'02	22'90	23'86	24'90	26'03	27'27					
88	28'64	30'14	31'82	33'69	35'80	38'19	40'92	44'07	47'74	52'08					
89	57'29	63'66	71'62	81'85	95'49	114'6	143'2	191'0	286'5	573'0					

NATURAL SECANTS.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
0	1'0000	0000	0000	0000	0000	0000	0001	0001	0001	0001	0	0	0	0	0
1	0002	0002	0002	0003	0003	0003	0004	0004	0005	0006	0	0	0	0	0
2	0006	0007	0007	0008	0009	0010	0010	0011	0012	0013	0	0	0	0	1
3	0014	0015	0016	0017	0018	0019	0020	0021	0022	0023	0	0	0	1	1
4	0024	0026	0027	0028	0030	0031	0032	0034	0035	0037	0	0	1	1	1
5	0038	0040	0041	0043	0045	0046	0048	0050	0051	0053	0	1	1	1	1
6	0055	0057	0059	0061	0063	0065	0067	0069	0071	0073	0	1	1	1	2
7	0075	0077	0079	0082	0084	0086	0089	0091	0093	0096	0	1	1	2	2
8	0098	0101	0103	0106	0108	0111	0114	0116	0119	0122	0	1	1	2	2
9	0125	0127	0130	0133	0136	0139	0142	0145	0148	0151	0	1	1	2	2
10	0154	0157	0161	0164	0167	0170	0174	0177	0180	0184	1	1	2	2	3
11	0187	0191	0194	0198	0201	0205	0209	0212	0216	0220	1	1	2	2	3
12	0223	0227	0231	0235	0239	0243	0247	0251	0255	0259	1	1	2	3	3
13	0263	0267	0271	0276	0280	0284	0288	0293	0297	0302	1	1	2	3	4
14	0306	0311	0315	0320	0324	0329	0334	0338	0343	0348	1	2	2	3	4
15	0353	0358	0363	0367	0372	0377	0382	0388	0393	0398	1	2	3	3	4
16	0403	0408	0413	0419	0424	0429	0435	0440	0446	0451	1	2	3	4	5
17	0457	0463	0468	0474	0480	0485	0491	0497	0503	0509	1	2	3	4	5
18	0515	0521	0527	0533	0539	0545	0551	0557	0564	0570	1	2	3	4	5
19	0576	0583	0589	0595	0602	0608	0615	0622	0628	0635	1	2	3	4	5
20	0642	0649	0655	0662	0669	0676	0683	0690	0697	0704	1	2	4	5	6
21	0711	0719	0726	0733	0740	0748	0755	0763	0770	0778	1	2	4	5	6
22	0785	0793	0801	0808	0816	0824	0832	0840	0848	0856	1	3	4	5	6
23	0864	0872	0880	0888	0896	0904	0913	0921	0929	0938	1	3	4	6	7
24	0946	0955	0963	0972	0981	0989	0998	1007	1016	1025	1	3	4	6	7
25	1034	1043	1052	1061	1070	1079	1089	1098	1107	1117	2	3	5	6	7
26	1126	1136	1145	1155	1164	1174	1184	1194	1203	1213	2	3	5	6	8
27	1223	1233	1243	1253	1264	1274	1284	1294	1305	1315	2	3	5	7	9
28	1326	1336	1347	1357	1368	1379	1390	1401	1412	1423	2	4	5	7	9
29	1434	1445	1456	1467	1478	1490	1501	1512	1524	1535	2	4	6	7	9
30	1547	1559	1570	1582	1594	1606	1618	1630	1642	1654	2	4	6	8	10
31	1666	1679	1691	1703	1716	1728	1741	1753	1766	1779	2	4	6	8	10
32	1792	1805	1818	1831	1844	1857	1870	1883	1897	1910	2	4	7	9	11
33	1924	1937	1951	1964	1978	1992	2006	2020	2034	2048	2	5	7	9	12
34	2062	2076	2091	2105	2120	2134	2149	2163	2178	2193	2	5	7	10	12
35	2208	2223	2238	2253	2268	2283	2299	2314	2329	2345	3	5	8	10	13
36	2361	2376	2392	2408	2424	2440	2456	2472	2489	2505	3	5	8	11	13
37	2521	2538	2554	2571	2588	2605	2622	2639	2656	2673	3	6	8	11	14
38	2690	2708	2725	2742	2760	2778	2796	2813	2831	2849	3	6	9	12	15
39	2868	2886	2904	2923	2941	2960	2978	2997	3016	3035	3	6	9	13	15
40	3054	3073	3093	3112	3131	3151	3171	3190	3210	3230	3	7	10	13	16
41	3250	3270	3291	3311	3331	3352	3373	3393	3414	3435	4	7	10	14	17
42	3456	3478	3499	3520	3542	3563	3585	3607	3629	3651	4	7	11	14	18
43	3673	3696	3718	3741	3763	3786	3809	3832	3855	3878	4	8	11	15	19
44	3902	3925	3949	3972	3996	4020	4044	4069	4093	4118	4	8	12	16	20

NATURAL SECANTS.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
45	1·4142	4167	4192	4217	4242	4267	4293	4318	4344	4370	4	8	13	17	21
46	4396	4422	4448	4474	4501	4527	4554	4581	4608	4635	4	9	13	18	22
47	4663	4690	4718	4746	4774	4802	4830	4859	4887	4916	5	10	14	19	23
48	4945	4974	5003	5032	5062	5092	5121	5151	5182	5212	5	10	15	20	25
49	5243	5273	5304	5335	5366	5398	5429	5461	5493	5525	5	10	16	21	26
50	5557	5590	5622	5655	5688	5721	5755	5788	5822	5856	6	11	17	22	28
51	5890	5925	5959	5994	6029	6064	6099	6135	6171	6207	6	12	18	23	29
52	6243	6279	6316	6353	6390	6427	6464	6502	6540	6578	6	12	19	25	31
53	6616	6655	6694	6733	6772	6812	6852	6892	6932	6972	7	13	20	26	33
54	7013	7054	7095	7137	7179	7221	7263	7305	7348	7391	7	14	21	28	35
55	7434	7478	7522	7566	7610	7655	7700	7745	7791	7837	7	15	22	30	37
56	7883	7929	7976	8023	8070	8118	8166	8214	8263	8312	8	16	24	32	40
57	8361	8410	8460	8510	8561	8612	8663	8714	8766	8818	8	17	25	34	43
58	8871	8924	8977	9031	9084	9139	9194	9249	9304	9360	9	18	27	36	45
59	9416	9473	9530	9587	9645	9703	9762	9821	9880	9940	10	19	29	39	49
60	2·0000	0061	0122	0183	0245	0308	0371	0434	0498	0562	10	21	31	42	52
61	0627	0692	0757	0824	0890	0957	1025	1093	1162	1231	11	22	33	45	56
62	1301	1371	1441	1513	1584	1657	1730	1803	1877	1952	12	24	36	48	61
63	2027	2103	2179	2256	2333	2412	2490	2570	2650	2730	13	26	39	52	66
64	2812	2894	2976	3060	3144	3228	3314	3400	3486	3574	14	28	43	57	71
65	3662	3751	3841	3931	4022	4114	4207	4300	4395	4490	15	31	46	62	77
66	4586	4683	4780	4879	4978	5078	5180	5282	5384	5488	17	34	50	68	84
67	5593	5699	5805	5913	6022	6131	6242	6354	6466	6580	18	37	55	74	92
68	6695	6811	6927	7046	7165	7285	7407	7529	7653	7778	20	40	61	81	1 1
69	7904	8032	8161	8291	8422	8555	8688	8824	8960	9099	22	44	67	89	111
70	9238	9379	9521	9665	9811	9957	$\overline{0}106$	$\overline{0}256$	$\overline{0}407$	$\overline{0}561$	25	49	74	98	123
71	3·0716	0872	1030	1190	1352	1515	1681	1848	2017	2188	27	55	82	110	137
72	2361	2535	2712	2891	3072	3255	3440	3628	3817	4009	31	61	92	122	153
73	4203	4399	4598	4799	5003	5209	5418	5629	5843	6060	35	69	104	138	173
74	6280	6502	6727	6955	7186	7420	7657	7897	8140	8387	39	78	118	157	196
75	8637	8890	9147	9408	9672	9939	$\overline{0}211$	$\overline{0}486$	$\overline{0}765$	$\overline{1}048$	45	90	134	180	224
76	4·1336	1627	1923	2223	2527	2837	3150	3469	3792	4121	52	104	156	207	260
77	4454	4793	5137	5486	5841	6202	6569	6942	7321	7706	61	121	182	242	303
78	8097	8496	8901	9313	9732	$\overline{0}159$	$\overline{0}593$	$\overline{1}034$	$\overline{1}484$	$\overline{1}942$	72	143	215	287	359
79	5·2408	2883	3367	3860	4362	4874	5396	5928	6470	7023	86	172	258	344	431
80	7588	8164	8751	9351	9963	$\overline{0}589$	$\overline{1}227$	$\overline{1}880$	$\overline{2}546$	$\overline{3}228$					
81	6·3925	4637	5366	6111	6874	7655	8454	9273	$\overline{0}112$	$\overline{0}972$					
82	7·1853	2757	3684	4635	5611	6613	7612	8700	9787	$\overline{0}905$					
83	8·2055	3238	4457	5711	7004	8337	9711	$\overline{1}129$	$\overline{2}593$	$\overline{4}105$					
84	9·5668	9·728	9·895	10·07	10·25	10·43	10·63	10·83	11·03	11.25					
85	11·47	11·71	11·95	12·20	12·47	12·75	13·03	13·34	13·65	13·99					
86	14·34	14·70	15·09	15·50	15·93	16·38	16·86	17·37	17·91	18·49					
87	19·11	19·77	20·47	21·23	22·04	22·93	23·88	24·92	26·05	27·29					
88	28·65	30·16	31·84	33·71	35·81	38·20	40·93	44·08	47·75	52·09					
89	57·30	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

Mean differences not sufficiently accurate.

LOGARITHMS OF NUMBERS.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2089	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3062	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

LOGARITHMS OF NUMBERS.

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTI-LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1107	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	8126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTI-LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
'50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
'51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
'52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
'53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
'54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
'55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
'56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
'57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
'59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
'60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
'61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
'63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
'64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
'65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
'66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
'67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
'68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
'69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
'70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
'71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
'72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
'73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
'74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
'76	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
'75	5754	5768	5781	5794	5808	5821	5834	5818	5861	5875	1	3	4	5	7	8	9	11	12
'77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
'78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
'79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
'80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
'81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
'82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
'83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
'84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
'85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
'86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
'87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
'88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
'89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
'90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
'91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
'92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
'93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
'94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
'95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
'96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
'97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
'98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
'99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

LOGARITHMIC SINES.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
0	—∞	7·2419	5429	7190	8439	9408	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$961					
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	21	42	63	84	104
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	8·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{0}$046	$\bar{0}$120	13	26	39	52	66
6	9·0192	0261	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

LOGARITHMIC SINES.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
45	9'8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	9'8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	9'8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	9'8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	9'8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	9'8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	9'8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	9'8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	9'9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	9'9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	9'9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	9'9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	9'9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	9'9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	9'9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	9'9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	9'9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	9'9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	9'9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	9'9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	9'9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	9'9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	9'9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	9'9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	9'9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	9'9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	9'9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	9'9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	9'9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	9'9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	9'9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	9'9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	9'9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	9'9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	9'9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	9'9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	9'9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	9'9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	9'9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	9'9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	9'9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	9'9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	9'9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	9'9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	9'9999	9999	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	0	0	0	0	0

LOGARITHMIC TANGENTS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
											1′	2′	3′	4′	5′
0	−∞	7·2419	5429	7190	8439	9409	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$962	*Mean* Differences not sufficiently accurate.				
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	65	81
5	8·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}$068	$\bar{0}$143	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6813	6863	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

LOGARITHMIC TANGENTS.

Degree	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
45	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	10 0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	10·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	10 2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	10·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	10·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	10·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	10·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	10·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	10·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	10·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	10·9784	9857	9932	$\overline{0}008$	$\overline{0}085$	$\overline{0}164$	$\overline{0}244$	$\overline{0}326$	$\overline{0}409$	$\overline{0}494$	13	26	40	53	66
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	65	81
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	*Mean* Differences not sufficiently accurate.				
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	11·7581	8038	8550	9130	9800	$\overline{0}591$	$\overline{1}561$	$\overline{2}810$	$\overline{4}571$	$\overline{7}581$					